1. Carefully define the following terms: \( \leq \) (for integers, as defined in Chapter 1), factorial, Associativity theorem (for propositions), Distributivity theorem (for propositions).

Let \( a, b \) be integers. We say that \( a \leq b \) if \( b - a \in \mathbb{N}_0 \). The factorial is a function from \( \mathbb{N}_0 \) to \( \mathbb{Z} \) (or \( \mathbb{N} \)), denoted by \(!\), defined by: \( 0! = 1 \) and \( n! = (n-1)! \cdot n \) (for \( n \geq 1 \)). The Associativity theorem says: Let \( p, q, r \) be propositions. Then \((p \land q) \land r \equiv p \land (q \land r)\) and also \((p \lor q) \lor r \equiv p \lor (q \lor r)\). The Distributivity theorem says: Let \( p, q, r \) be propositions. Then \( p \land (q \lor r) \equiv (p \land q) \lor (p \land r)\) and also \( p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)\).

2. Carefully define the following terms: Addition semantic theorem, Contrapositive Proof theorem, Direct Proof, converse.

The Addition semantic theorem states that for any propositions \( p, q \), we have \( p \vdash p \lor q \). The Contrapositive Proof theorem states that for any propositions \( p, q \), if \((\neg q) \vdash (\neg p)\) is valid, then \( p \rightarrow q \) is \( T \). The Direct Proof theorem states that for any propositions \( p, q \), if \( p \vdash q \) is valid, then \( p \rightarrow q \) is \( T \). The converse of conditional proposition \( p \rightarrow q \) is \( q \rightarrow p \).

3. Let \( a, b \) be odd. Prove that \( 4a - 3b \) is odd.

Because \( a \) is odd, there is integer \( c \) with \( a = 2c + 1 \). Because \( b \) is odd there is integer \( d \) with \( b = 2d + 1 \). Now, 
\[
4a - 3b = 4(2c + 1) - 3(2d + 1) = 8c + 4 - 6d - 3 = 2(4c - 3d) + 1.
\]
Because \( 4c - 3d \) is an integer, \( 4a - 3b \) is odd.

4. Suppose that \( a | b \). Prove that \( a(4a - 3b) \).

Because \( a | b \), there is integer \( c \) with \( b = ca \). Now, \( 4a - 3b = 4a - 3(ca) = a(4 - 3c) \). Because \( 4 - 3c \) is an integer, \( a(4 - 3b) \).

5. Simplify \( (\neg(p \rightarrow q) \lor (p \rightarrow r)) \) to use only \( \neg, \lor, \land \), and to have only basic propositions negated.

Applying De Morgan’s law, we get \((\neg(p \rightarrow q) \lor (p \rightarrow r))\). Applying a theorem from the book (2.16), we get \((p \land \neg q) \lor (p \land \neg r)\). Applying associativity and commutativity of \( \land \) several times, we get \((p \land \neg q) \lor (p \land \neg r)\). Applying a theorem from the book (2.7), we get \( p \land \neg q \lor \neg r \).

6. Without truth tables, prove the Constructive Dilemma theorem, which states: Let \( p, q, r, s \) be propositions. \( p \rightarrow q, r \rightarrow s, p \lor r \rightarrow q \lor s \).

Because \( p \lor r \) is \( T \) (by hypothesis), we have two cases: \( p \) is \( T \) or \( r \) is \( T \). If \( p \) is \( T \), we apply modus ponens to \( p \rightarrow q \) to conclude \( q \). We then apply addition to get \( q \lor s \). If instead \( r \) is \( T \), we apply modus ponens to \( r \rightarrow s \) to conclude \( s \). We apply addition to get \( q \lor s \). In both cases \( q \lor s \) is \( T \).

7. State the Conditional Interpretation theorem, and prove it using truth tables.

The CI theorem states:

\[
\begin{array}{ccc|c|c|}
 & p & q & \neg p & q \lor \neg p \\
\hline T & T & T & F & T \\
F & T & F & F & F \\
T & F & T & T & T \\
F & F & T & T & T \\
\end{array}
\]

Proof: The third and fifth columns in the truth table at right, as shown, agree. Hence \( p \rightarrow q \equiv q \lor \neg p \).

8. Let \( x \in \mathbb{R} \). Suppose that \( [x] = [x] \). Prove that \( x \in \mathbb{Z} \).

First, \( [x] \leq x \) by definition of floor. Second, \( x \leq [x] \) by definition of ceiling. But since \( [x] = [x] \), in fact \( x \leq [x] \). Combining with the first fact, \( x = [x] \). Since \( [x] \) is an integer, so is \( x \).

9. Prove or disprove: For arbitrary propositions \( p, q \), \( (p \downarrow q) \rightarrow (p \uparrow q) \) is a tautology.

Since the fifth column in the truth table at right, as shown, is all \( T \), the proposition \( (p \downarrow q) \rightarrow (p \uparrow q) \) is indeed a tautology.

10. Prove or disprove: For arbitrary \( x \in \mathbb{R} \), if \( x \) is irrational then \( 2x - 1 \) is irrational.

The statement is true, we provide a contrapositive proof. Suppose that \( 2x - 1 \) is rational. Then there are integers \( a, b \), with \( b \) nonzero, such that \( 2x - 1 = \frac{a}{b} \). We have \( 2x = \frac{a}{b} + 1 = \frac{a+b}{b} \), and \( x = \frac{a+b}{2b} \). Now, \( a + b, 2b \) are integers, and \( 2b \) is nonzero, so \( x \) is rational.