Essential Concepts:

Integration by Parts:

\[ \int x \cos (5x) \, dx \quad \int \ln (x) \, dx \quad \int x e^{-x/2} \, dx \quad \int x e^{-4x} \, dx \quad \int x^2 \ln (x) \, dx \]

\[ \int x \sin (x/3) \, dx \quad \int \ln (x) \, dx \quad \int x e^{-x/2} \, dx \quad \int \arcsin (x) \, dx \quad \int x e^{-x/3} \, dx \]

\[ \int \cosh (\frac{x}{4}) \, dx \quad \int x \cos (\frac{x}{2}) \, dx \quad \int \cosh (5x) \, dx \]

(yes, the same integral has appeared 3 times)

4 (10 pts.) Determine whether the improper integral

\[ \int_{-\infty}^{\infty} x e^x \, dx \]

converges or diverges, and its value in case it converges.

Partial Fractions:

\[ \int \frac{x - 19}{(x - 3)(x + 1)} \, dx \quad \int \frac{4x^2 + 7x - 3}{(x + 1)^2 (x - 2)} \, dx \quad \int \frac{x - 23}{x^3 + 3x - 10} \, dx \quad \int \frac{7x + 11}{(x + 3)(x - 2)} \, dx \]

\[ \int \frac{x + 3x}{x^2 - 3x - 10} \, dx \quad \int \frac{x - 11}{x^2 - 3x - 2} \, dx \quad \int \frac{4x^2 + 2x + 19}{(x - 2)(x^2 + 9)} \, dx \quad \int \frac{3x + 14}{x^3 + x - 8} \, dx \]

4. (5 pts) Find the form of the partial fraction decomposition for the expression

\[ \frac{3x^3 + 13x^2 + 7x = 17x + 14}{(x + 2)(x^2 + 1)(x + 3)} \]

You do not need to solve for the coefficients.

Improper Integrals:

4 (10 pts) Determine whether the improper integral

\[ \int_{-\infty}^{\infty} x e^x \, dx \quad \int_{0}^{\infty} 4^{-x} \, dx \]

converges or diverges, and its value in case it converges.

4. (10 pts.) Determine whether the improper integral \( \int_{0}^{\infty} \frac{1}{x^4 + 4} \, dx \) converges or diverges, and its value in the case of convergence.

(b) (5 pts) Determine whether the improper integral

\[ \int_{0}^{1} \frac{\ln (x)}{x^{1/3}} \, dx \]

converges or diverges and its value in the case of convergence.

3. (10 pts.) Use the integral test to determine whether the infinite series

\[ \sum_{n=1}^{\infty} \frac{1}{n \ln^2 (n)} \]

converges or diverges.
5. (5 pts.) Determine whether the improper integral
\[ \int_{2}^{4} \frac{1}{x-2} \, dx \]
converges or diverges, and the value of the improper integral in case of convergence.

4 (5 pts.) Determine whether the improper integral
\[ \int_{0}^{\infty} \frac{x}{(x^2 + 1)^2} \, dx \]
converges or diverges, and its value in case it converges (express your response in a simplified form).

4 (5 pts.) Determine whether the improper integral
\[ \int_{0}^{\infty} \frac{1}{x^2 + 4x^2} \, dx \]
converges or diverges, and its value in case it converges.

(Absolute) Ratio Test:

6 (7 pts.) Use the ratio test to determine whether the infinite series
\[ \sum_{n=1}^{\infty} (-1)^n \frac{n!}{10^n} \]
converges absolutely or diverges.

10. (5 pts.) Determine whether the infinite series
\[ \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} \]
converges absolutely or conditionally.

8 (5 pts.) Determine whether the infinite series
\[ \sum_{n=0}^{\infty} \frac{1}{n!} \]
converges absolutely, converges conditionally or diverges. Justify your response.

4 (10 pts.) Use the ratio test to determine whether the infinite series
\[ \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^n}{n!} \]
converges or diverges.

Determine whether the infinite series
\[ \sum_{n=1}^{\infty} \frac{(n^2)!}{n!} \]
converges or diverges.
10 (5 pts.) Determine whether the infinite series
\[ \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n!} \]
converges absolutely, converges conditionally or diverges.

**Interval of Convergence (absolute ratio test):**

9 (8 pts.) Determine the radius of convergence and the open interval of convergence of the power series
\[ \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{4^n} (x - 2)^n \]
(You need not worry about the endpoints of the interval).

7 (10 pts.) Determine the radius of convergence and the open interval of convergence of the power series
\[ \sum_{n=1}^{\infty} \frac{(x - 4)^n}{n^2} \]
(You need not investigate the series at the endpoints of the interval.)

7 (10 pts.) Determine the radius of convergence and the open interval of convergence of the power series
\[ \sum_{n=0}^{\infty} (-1)^n \frac{n}{2^n} (x - 4)^n \]
(you need not investigate the series at the endpoints of the interval)

8. (10 pts.) Determine the radius of convergence and the open interval of convergence of the power series
\[ \sum_{n=1}^{\infty} \frac{(x - 1)^n}{2^n n^{3/2}} \]
(You need not investigate the series at the endpoints of the interval.)

6 (10 pts.) Determine the radius of convergence and the open interval of convergence of the power series
\[ \sum_{n=1}^{\infty} \frac{(x - 2)^n}{2^n n^{1/2}} \]
(You need not investigate the series at the endpoints of the interval.)

12. (10 pts.) Determine the radius of convergence and the open interval of convergence of the power series
\[ \sum_{n=1}^{\infty} \frac{(x - 3)^n}{4^n n^{1/2}} \]
(You need not investigate the series at the endpoints of the interval.)

12 (5 pts.) Determine the radius of convergence and the open interval of convergence of the power series
\[ \sum_{n=0}^{\infty} (-1)^n \frac{2^n}{n^2} (x - 4)^n \]
(You need not investigate the series at the endpoints of the interval.)

12 (10 pts.) Determine the radius of convergence and the open interval of convergence of the power series
\[ \sum_{n=0}^{\infty} (-1)^n \frac{n^2}{2^n} (x - 4)^n \]
(You need not investigate the series at the endpoints of the interval.)
12. (10 pts.) Determine the radius of convergence and the open interval of convergence of the power series:
\[ \sum_{n=1}^{\infty} \frac{\sqrt{n}}{3n^2} (x+6)^n. \]
(You need not investigate the series at the endpoints of the interval.)

**Taylor Series:**

10. We have
\[ \arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \]
\[ (\arctan(x) = \tan^{-1}(x)). \]
a) (5 pts.) Determine
\[ \lim_{x \to 0} \frac{\arctan(x^2) - x^2}{x^2} \]
by using the Maclaurin series of \(\arctan(x)\) (Do not use L'Hôpital's rule).
b) (5 pts.) Let
\[ F(x) = \int_0^x \frac{\arctan(t^2)}{t^2} \, dt. \]
Determine the first three terms of the Maclaurin series for \(F(x)\).

8. (10 pts.) Given that \(e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\), determine the Maclaurin series for
\[ F(x) = \int_0^x \frac{e^t - 1 - t}{t^2} \, dt. \] (Display the first 4 terms and the term involving \(x^4\). You may leave your answer in terms of the factorial.)

\[ F(x) = \int_0^x \frac{1}{1 + t^4} \, dt. \]
Determine the Maclaurin series for \(F\) up to the term that has \(x^4\).

9. (10 pts.) Given that
\[ \arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \quad -1 \leq x \leq 1 \]
determine the first four nonzero terms of the Maclaurin series for
\[ \frac{d}{dx} \arctan(x^2) \]
Simplify as much as possible.

Determine the Maclaurin polynomial of order 3 for \((x + 1)^{1/2}\).

8. (10 pts.) Let
\[ F(x) = \int_0^x \frac{1}{1 - t^2} \, dt. \]
Determine the Maclaurin series for \(F\) (display the first 3 terms and the term involving \(x^{2n+1}\)).

13. (5 pts.) Determine the Maclaurin series of
\[ f(x) = \frac{1}{1 + x^2} \]
and the open interval of convergence of the series (Display the first 4 terms and the term that involves \(x^{2n}\) for an arbitrary positive integer \(n\)).

Hint: Think of the geometric series.

14. (5 pts.) Let \(f(x) = \sin(x)\). Determine the first 4 terms of the Taylor series of \(f\) in powers of \((x - \pi/6)\).
13 (5 pts.) Let \[ F(x) = \int_0^x t^2 e^{-t^2} dt. \]

Determine the first 4 nonzero terms of the Maclaurin series for \( F \) and the coefficient of \( x^{2n+3} \), where \( n \) is an arbitrary nonnegative integer.

Hint: Start with the Maclaurin series for the natural exponential function.

Let
\[
f(x) = \begin{cases} 
\frac{\sin(x) - x}{x^3} & \text{if } x \neq 0, \\
-\frac{1}{6} & \text{if } x = 0.
\end{cases}
\]

Determine the first 3 nonzero terms of the Maclaurin series for \( f \) and the coefficient of \( x^{2n-2} \), where \( n \) is an arbitrary positive integer.

Hint: Start with the Maclaurin series for sine.

13 (5 pts.) Let \( f(x) = \ln(x) \). Determine the part of the Taylor series for \( f \) based at 2 up to the term that has \( (x - 2)^3 \).

14. (5 pts.) Let \( f(x) = \sin(x) \). Determine the first 4 terms of the Taylor series of \( f \) in powers of \( (x - \pi/6) \).

13 (5 pts.) Let \( f(x) = \sin(x) \). Determine the part of the Taylor series for \( f \) based at \( \pi/2 \) up to the term that has \( (x - \pi/6)^2 \).

14 (5 pts.) Given that \[
\frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \ldots
\]

Determine the Maclaurin series for \( \arcsin(x) \) up to the term that has \( x^7 \).

**Differential Equations:**

7. (10 pts.) Make use of the technique of an integrating factor to determine the solution of the initial value problem
\[
\frac{dy}{dt} + 2ty(t) = 2te^{-t^2}, \quad y(2) = 1.
\]

7. (10 pts.) Make use of the technique of an integrating factor to determine the solution of the initial value problem
\[
\frac{dy}{dt} + \frac{y(t)}{t} = \sin(t), \quad y(\pi) = 3
\]

Assume that \( t > 0 \).

7. (10 pts.) Make use of the technique of an integrating factor to determine the solution of the initial value problem
\[
\frac{dy}{dt} + y(t) = e^{-t}, \quad y(0) = 2.
\]

Determine the solution of the initial value problem
\[
\frac{dy}{dt} = -\frac{1}{4}y(t) + 3t, \quad y(4) = 2.
\]
Determine the solution of the initial value problem \[ \frac{dy}{dx} = ye^{-x}, \quad y(0) = 2 \]

Determine the solution of the initial value problem \[ \frac{dy}{dt} = \frac{ty^2}{\sqrt{1 + t^2}}, \quad y(0) = 3 \]

Determine the solution of the initial value problem \[ \frac{dy}{dx} = \frac{y^2}{1 + x^2}, \quad y(1) = -\frac{2}{e} \]

Determine the solution of the initial value problem \[ \frac{dy}{dt} = \sin(t)y^2, \quad y(\pi) = 4 \]

**Polar Coordinates:**

9. Let \[ r = f(\theta) = 1 + 2\cos(\theta). \]
   
   a) (5 pts.) Sketch the graph of \( r = f(\theta) \) in the Cartesian \( \theta r \)-plane on the interval \([0, 2\pi]\). Indicate the values of \( \theta \) at which \( f(\theta) = 0 \) and the points at which \( f \) attains a maximum or minimum value.

   b) (10 pts.) Sketch the graph of \( r = f(\theta) \) as a polar equation in the \( xy \)-plane (i.e., \( x = r\cos(\theta), \ y = r\sin(\theta) \)).

11. Let \( f(\theta) = 2 + 4\cos(\theta) \).
    
    a) (5 pts.) Sketch the graph of \( r = f(\theta) \) in the Cartesian \( \theta r \)-plane on the interval \([0, 2\pi]\). Indicate the values of \( \theta \) at which \( f(f(\theta)) = 0 \).

    b) (8 pts.) Sketch the graph of \( r = f(\theta) \) in the Cartesian \( xy \)-plane if \( r \) and \( \theta \) are polar coordinates, i.e., \( x = r\cos(\theta), \ y = r\sin(\theta) \).

\[ r = f(\theta) = 2 - \cos(\theta) \]

   a) (5 pts.) Sketch the graph of \( r = f(\theta) \) the Cartesian \( \theta r \)-plane on the interval \([0, 2\pi]\).

   b) (10 pts.) Sketch the graph of \( r = f(\theta) \) in the Cartesian \( xy \)-plane if \( r \) and \( \theta \) are polar coordinates (i.e., \( x = r\cos(\theta), \ y = r\sin(\theta) \)).

\[ r = f(\theta) = 3\sin(2\theta) \]

   a) (5 pts.) Sketch the graph of \( r = f(\theta) \) in the Cartesian \( \theta r \)-plane on the interval \([0, 2\pi]\). Indicate the values of \( \theta \) at which \( f(\theta) = 0 \) and the points at which \( f \) attains a maximum or minimum value.

   b) (5 pts.) Sketch the graph of \( r = f(\theta) \) as a polar equation in the \( xy \)-plane (i.e., \( x = r\cos(\theta), \ y = r\sin(\theta) \)).

\[ r = f(\theta) = 1 - 2\cos(\theta) \]

   a) (5 pts.) Sketch the graph of \( r = f(\theta) \) in the Cartesian \( \theta r \)-plane on the interval \([0, 2\pi]\). Indicate the values of \( \theta \) at which \( f(\theta) = 0 \) and the points at which \( f \) attains a maximum or minimum value.

   b) (10 pts.) Sketch the graph of \( r = f(\theta) \) as a polar equation in the \( xy \)-plane (i.e., \( x = r\cos(\theta), \ y = r\sin(\theta) \)).

\[ r = f(\theta) = 1 - 2\cos(\theta) \]

   a) (6 pts.) Sketch the graph of \( r = f(\theta) \) in the Cartesian \( \theta r \)-plane on the interval \([0, 2\pi]\). Indicate the values of \( \theta \) at which \( f(\theta) = 0 \) and the points at which \( f \) attains a maximum or minimum value.

   b) (5 pts.) Sketch the graph of \( r = f(\theta) \), where \( 0 \leq \theta \leq 2\pi \), as a polar equation in the \( xy \)-plane (i.e., \( x = r\cos(\theta), \ y = r\sin(\theta) \)).
\[ r = f(\theta) = \cos(2\theta). \]

a) (5 pts.) Sketch the graph of \( r = f(\theta) \) in the Cartesian \( \theta r \)-plane on the interval \([0, 2\pi]\).
Indicate the values of \( \theta \) at which \( f(\theta) = 0 \) and the points at which \( f \) attains a local maximum or minimum value.

b) (5 pts.) Sketch the graph of \( r = f(\theta) \), where \( 0 \leq \theta \leq 2\pi \), as a polar equation in the \( xy \)-plane (i.e., \( x = r \cos(\theta), y = r \sin(\theta) \)).

\[ r = f(\theta) = 2 - 2 \sin(\theta). \]

a) (5 pts.) Sketch the graph of \( r = f(\theta) \) in the Cartesian \( \theta r \)-plane on the interval \([0, 2\pi]\).
Indicate the values of \( \theta \) at which \( f(\theta) = 0 \) and the points at which \( f \) attains a local maximum or minimum value.

b) (5 pts.) Sketch the graph of \( r = f(\theta) \), where \( 0 \leq \theta \leq 2\pi \), as a polar equation in the \( xy \)-plane (i.e., \( x = r \cos(\theta), y = r \sin(\theta) \)).

\[ r = f(\theta) = \cos(2\theta). \]

Important Concepts:

**Trigonometric Integrals:**

\[ \int \cos^2(x) \sin^2(x) \, dx \quad \int \sin^3(4x) \, dx \quad \int \sin^2(x) \cos^2(x) \, dx \quad \int \sin^3(3x) \, dx \quad \int \cos^3\left(\frac{x^2}{4}\right) \, dx. \]

6. (5 pts.) Determine the volume of the solid that is obtained by revolving the graph of

\[ f(x) = \cos\left(\frac{x^2}{2}\right) \]

on the interval \([-\pi, \pi]\) about the \( x \)-axis.

**Limit Comparison Test:**

7. (10 pts.) Determine whether the infinite series

\[ \sum_{n=1}^{\infty} \frac{n^3}{4n^4 - 19n + 2} \]

converges or diverges.

Hint: Use the limit comparison test.

6. (10 pts.) Determine whether the infinite series

\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + 3} \]

converges absolutely, converges conditionally, or diverges.

11. (10 pts.) Determine whether the infinite series

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2 + 1} \]

converges absolutely, converges conditionally or diverges.

11. (10 pts.) Determine whether the infinite series

\[ \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(n^2 + 1)^{3/2}} \]

converges absolutely, converges conditionally or diverges.
Integral Comparison Test:

4 (5 pts.) Does the improper integral
\[ \int_1^\infty \frac{\sin^2(x)}{x^3} \, dx \]
converge or diverge? Justify your response.
Hint: Make use of a comparison theorem.

5 (5 pts.) Use the comparison test to determine whether the improper integral
\[ \int_0^1 e^{x^2} \frac{1}{x^{3/2}} \, dx \]
converges or diverges (if you claim that the integral converges, you do not have to evaluate the value of the integral).

5 (5 pts.) Use a comparison test to determine whether the improper integral
\[ \int_1^\infty \frac{\sin^2(x) \, dx}{e^x} \]
converges or diverges.

5 (5 pts.) Use a comparison test to determine whether the improper integral
\[ \int_1^\infty e^{-x^2} \frac{1}{x^4} \, dx \]
converges or diverges. You need not determine the value of the improper integral in case of convergence.

Volumes:

6. (5 pts.) Determine the volume of the solid that is obtained by revolving the graph of
\[ f(x) = \cos \left( \frac{x}{\pi} \right) \]
on the interval \([-\pi, \pi]\) about the x-axis.

6 (10 pts.) Determine the volume of the solid that is obtained by revolving the region between the graph of
\[ f(x) = \frac{1}{x^3 + 4} \]
and the interval \([0, 2]\) about the vertical axis.

6 (10 pts.) Determine the volume of the solid that is obtained by revolving the region between the graph of
\[ f(x) = \sqrt{\frac{x}{4}} \]
and the interval \([\sqrt{4} - 4, \sqrt{4} - 4]\) about the x-axis.

6 (10 pts.) Determine the area of the surface that is obtained by revolving the graph of
\[ f(x) = x^2 \]
and the interval \([0, 2]\) about the x-axis.
Integral test:

11 (10 pts.) Determine whether the infinite series

\[ \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n \ln^n(n)} \]

converges absolutely, converges conditionally or diverges.

5 (10 pts.) Use the integral test to determine whether the infinite series

\[ \sum_{n=2}^{\infty} \frac{n^2}{n^4 + 4} \]

converges or diverges (you need not justify the applicability of the test).

11 (10 pts.) Determine whether the infinite series

\[ \sum_{n=2}^{\infty} \frac{(-1)^{n-1} \ln(n)}{n} \]

converges absolutely, converges conditionally or diverges.

6. (10 pts.) Use the integral test to determine if the infinite series

\[ \sum_{n=1}^{\infty} ne^{-n^2} \]

converges or diverges. (You need not justify the applicability of the test.)