On the monotonicity of the number of positive entries in nonnegative five element matrix powers

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Abstract

Let A be an $m \times m$ square matrix with nonnegative entries and let $F(A)$ denote the number of positive entries in A. We consider the adjacency matrix A with a corresponding digraph with m vertices. $F(A)$ corresponds to the number of directed edges in the corresponding digraph. We consider conditions on A to make the sequence $\{F(A^n)\}_{n=1}^{\infty}$ monotonic. Monotonicity is known for $F(A) \leq 4$ (except for 3 non-monotonic cases) or $F(A) \geq m^2 - 2m + 2$; we extend this to $F(A) = 5$.

Keywords: nonnegative matrix; power; monotonicity; directed graph; adjacency matrix

Mathematical Subject Classification 2020: 15B34, 15B48, 05C20

1 Introduction

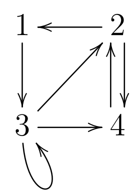
Nonnegative matrices are matrices with nonnegative real entries. Nonnegative matrices are valuable to study as they can be applied to fields such as probability, economics, and combinatorics (see [1]). We define F to be a function from the nonnegative square matrices to the integers that counts the number of positive entries in nonnegative square matrices. Then for any nonnegative matrix A, we can classify the sequence $\{F(A^n)\}_{n=1}^{\infty}$ as non-monotonic, monotonically increasing, monotonically decreasing, or constant.

It is clear to see that the value of each positive matrix value does not give any greater insight into the question of monotonicity. Thus, we can define our matrices to be of Boolean propositions as defined in [4]. These propositions can be one of two elements, unity and zero, with the operations presented to the right for reference.

+	0	1
0	0	1
1	1	1

×	0	1
0	0	0
1	0	1

For brevity, we can call these square matrices of Boolean propositions 0-1 matrices. There is a correspondence between a directed graph and a 0-1 matrix, known as an "adjacency matrix". It is common to observe adjacency matrices of digraphs (see [8]), with adjacency matrices already conforming to the Boolean propositions seen in 0-1 matrices. Adjacency matrices have many applications, examples including when studying strongly regular graphs and two-graphs (see [7]). Another application of the adjacency matrices we are studying is that adjacency matrices of strongly connected graphs are irreducible, and so the Perron-Frobenius Theorem can be related to these matrices (see [3]).



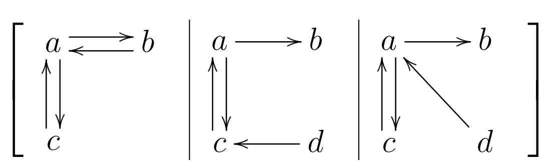
$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

An adjacency matrix has a 1 in its i^{th} row and j^{th} column if there is a directed edge from vertex i to vertex j. Since we are examining adjacency matrices that are 0-1 matrices, we need not consider digraphs with repeated edges from one vertex to another. To the left is an example of this correspondence.

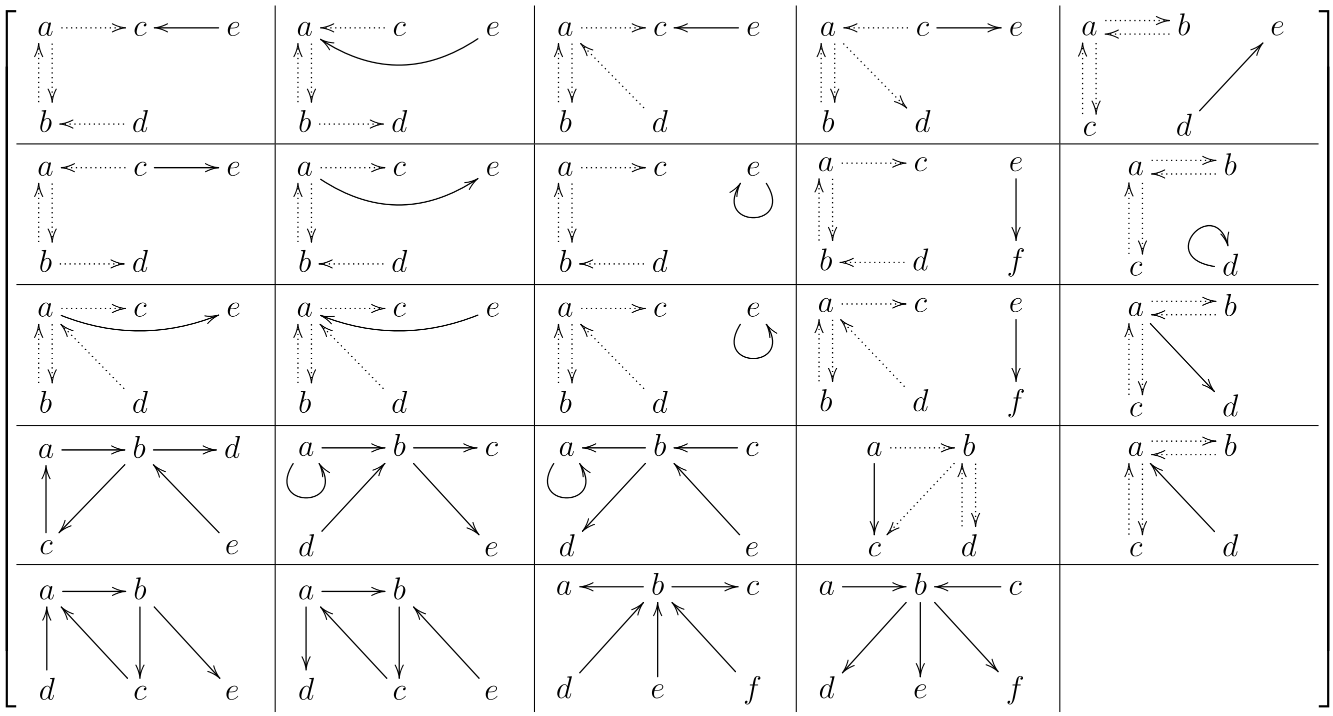
We note that the (i, j) entry of an adjacency matrix A^k shows whether or not there is at least 1 directed path of length k from vertex i to vertex j (if more than one path of length k exists, the adjacency matrix entry is unity regardless). So, we have that $F(A^k)$ is equal to the number of edges in the digraph corresponding to the adjacency matrix A^k. Then, we can instead observe the number of directed edges in a digraph composed with itself k times instead of directly computing A^k and counting the number of positive entries. Thus, we only need examine directed graphs with 5 edges (no repeated edges) and verify whether the number of edges as the digraph is composed with itself repeatedly is monotonic or not. We note that adjacency matrices of undirected graphs are symmetric, but as our goal is to study nonnegative square matrices, it is more useful to study the adjacency matrices of directed graphs that may or may not be symmetric.

In a brief digression, we note how if we have a digraph that has disjoint parts, then we can observe this corresponds to a block diagonal adjacency matrix, denote it B. As we find B^k for any $k \in \mathbb{N}$, we note that the entries in each block do not affect the entries in another block. As such, each of the digraph's disjoint parts, which correspond to a block in B, will never develop edges that connect the disjoint parts for any B^k. This means if we have that each of the disjoint parts of the digraph, corresponding to blocks, have a monotonically increasing number of edges, then the adjacency matrix B will be monotonically increasing. We reach a similar conclusion if the parts of the digraphs all have a monotonically decreasing number of edges. So if we have a digraph case with disjoint parts that are all monotonically increasing/decreasing we can easily determine the whole digraph to be monotonic and not include it in our list of cases in this paper's Section 3. However, if we have a digraph case with disjoint parts that have some being monotonically increasing and others being monotonically decreasing, then we include these cases in our work as we can reach no such conclusion.

In [2] and [9]; Xie, Brower, and Ponomarenko proved that for any $m \times m$ 0-1 matrix A, if $F(A) \leq 4$ or $F(A) \geq m^2 - 2m + 2$ then the sequence $\{F(A^n)\}_{n=1}^{\infty}$ is monotonic, except for 3 non-monotonic cases. To the right are the only non-monotonic cases.

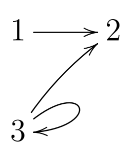


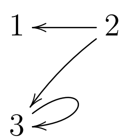
Our results allow us to conclude that all 0-1 matrices A, with $F(A) = 5$, are monotonic, except for the following cases shown below. Dotted edges demonstrate that some of these non-monotonic cases for $F(A) = 5$ have subsets that form a non-monotonic case from previous work.



In the second section, we establish some important terminology and theorems that will be useful in proving the monotonicity of the number of positive entries in nonnegative five element matrix powers.

2 Theorems

A  $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

A^T  $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

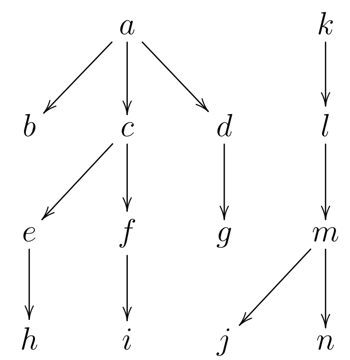
To begin, consider the digraph and corresponding adjacency matrix A to the left. We have $F(A) = 3$, and through direct calculation we have that the sequence $\{F(A^k)\}_{k=2}^{\infty} = 2$. Now, let us consider A^T. Similarly, we have through direct calculation that $F(A^T) = 3$ and $\{F((A^T)^k)\}_{k=2}^{\infty} = 2$. We also observe that graphically, the digraph corresponding to A has the direction of its directed edges switched when finding the digraph corresponding to A^T. With this in mind, we want to prove that for any $k \in \mathbb{N}$, $F(A^k) = F((A^T)^k)$ when considering adjacency matrices. This would mean that if we can prove, either through calculation or a theorem later in this paper, that a certain kind of digraph corresponding to an adjacency matrix has the sequence $\{F(A^k)\}_{k=1}^{\infty}$ being monotonic, then we can say the digraphs where the direction of the directed edges are reversed are also monotonic. The following theorem proves just that.

Theorem 2.1. *Let A be a square 0-1 matrix. Then $\forall k \in \mathbb{N}$, $F(A^k) = F((A^T)^k)$.*

Proof. Suppose we have a square 0-1 matrix A. Letting $k \in \mathbb{N}$, we first inductively prove $(A^k)^T = (A^T)^k$. (Base case) Suppose $k = 1$. We have $(A^1)^T = A^T = (A^T)^1$, as desired. (Inductive case). Assume $(A^k)^T = (A^T)^k$. We have $(A^{k+1})^T = (A^kA)^T = A^T(A^k)^T = A^T(A^T)^k = (A^T)^{k+1}$, as desired. Now consider A^k. Let $F(A^k) = p$, where $p \in \mathbb{N}_0$. Finding $(A^k)^T$, we have that $F((A^k)^T)$ will also equal p. Then, by applying the previous finding, we have $F(A^k) = F((A^k)^T) = F((A^T)^k)$, as desired. □

We provide the definition below to provide a shorthand when discussing a digraph corresponding to an adjacency matrix.

Definition 2.2. *Suppose A is an adjacency matrix corresponding to a digraph. We call this corresponding digraph the adjacency digraph of A, and denote it D_A.*



We now define in-forests and out-forests as they will be observed in subsequent theorems. To the left is an example of an out-forest with two connected components (i.e. out-trees).

Definition 2.3. *A rooted tree is a tree that has had a vertex assigned to be the root. A directed rooted tree is a rooted tree whose edges are assigned an orientation, either away from or towards the root. When a directed rooted tree has an orientation away from the root, we call this an out-tree. When a directed rooted tree has an orientation towards the root, we call this an in-tree. We call a disjoint collection of out-trees (similarly in-trees) an out-forest (similarly an in-forest). Each connected component of an out-forest (similarly an in-forest) is an out-tree (similarly an in-tree).*

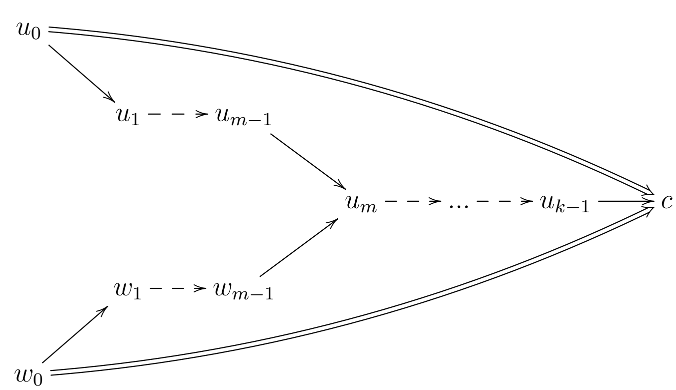
The theorems below will together show if D_A is an adjacency digraph that is an out-forest or an in-forest, then we have the sequence $\{F(A^k)\}_{k=1}^{\infty}$ will be monotonically decreasing.

Theorem 2.4. *A forest with k trees, on n vertices, has exactly $n - k$ edges.*

Proof. (Being a well-known theorem, we find it in most literature involving graph theory, see [5]) □

Theorem 2.5. *Let D_A be an adjacency digraph that is an out-forest with n vertices and 1 connected component (i.e. an out-tree) Let $p, k \in \mathbb{N}$. Then D_{A^k} will also be an out-forest with p connected components, and $F(A) = n - 1 \geq n - p = F(A^k)$.*

(The proof refers to the digraph here: D_A [Normal arrows], D_{A^k} [Thicker arrows])



Proof. Let D_A be an adjacency digraph that is an out-forest with n vertices and 1 connected component (i.e. an out-tree). By Theorem 2.4, we have $F(A) = n - 1$. In D_A we know every vertex has an indegree of either 1 or 0, and an arbitrary outdegree.

Now consider an arbitrary vertex c in D_A. We argue by way of contradiction. Suppose c in D_{A^k} has an indegree of more than 1, for arbitrary $k \in \mathbb{N}$. This means there are at least 2 paths of length k in D_A from some arbitrary vertices leading to c. Let the vertices in one path be denoted by the sequence $u_0, u_1, ..., u_k$, such that there is a directed edge from u_i to u_j, where $i = j - 1$, and $u_k = c$. Similarly, let the vertices in another path be denoted by the sequence $\{w_b\}_{b=0}^{k}$, such that there is a directed edge from w_i to w_j, where $i = j - 1$, and $w_k = c$. Let m be minimal where $u_m = w_m$. Consider the digraph above.

We observe that the vertex $u_m = w_m$ will have an indegree of 2 in D_A. This is a contradiction since we said D_A is an out-tree with all vertices having an indegree of 1 or 0. Hence there do not exist any vertices in D_{A^k} with an indegree of at least 2, so all vertices in D_{A^k} have an indegree of 1 or 0. Thus, we have shown D_{A^k} is an out-forest with n vertices. Since D_{A^k} is an out-forest, then clearly it has at least 1 connected component. Denote the number of connected components in D_{A^k} as $p \in \mathbb{N}$. By applying Theorem 2.4 twice, we have $F(A^k) = n - p \leq n - 1 = F(A)$, as desired. □

Corollary 2.6. *Let D_A be an adjacency digraph that is an out-forest with n vertices and $p \in \mathbb{N}$ connected components. Then D_{A^k} will also be an out-forest with $p' \in \mathbb{N}$ connected components, where $p' \geq p$, and $F(A) = n - p \geq n - p' = F(A^k)$.*

Proof. Let D_A be an adjacency digraph that is an out-forest with n vertices and $p \in \mathbb{N}$ connected components. Applying Theorem 2.4, we know $F(A) = n - p$. We observe that each connected component in D_A is disjoint from the other connected components by definition. So each of the p connected components (aka out-trees) in D_A can be handled separately by applying the idea of block diagonal matrices. Let D_{A_i} correspond to the *ith* out-tree in D_A, where $D_{A_1} \cup D_{A_2} \cup ... \cup D_{A_p} = D_A$. For each of these individual D_{A_i}'s which are out-trees, we say they each have n_i vertices, such that $n_1 + n_2 + ... + n_p = n$. By applying Theorem 2.5, we know for any D_{A_i}, $F(A_i^k) = n_i - b_i \leq n_i - 1 = F(A_i)$, where $b_i \in \mathbb{N}$ denotes the number of out-trees in $D_{A_i^k}$, such that $b_i \geq 1$. By combining this information for all D_{A_i}'s, we have $F(A^k) = \sum_{i=1}^{p} F(A_i^k) = \sum_{i=1}^{p} n_i - \sum_{i=1}^{p} b_i = n - p' \leq n - p = F(A)$, where $p' = \sum_{i=1}^{p} b_i \geq p$, as desired. □

Theorem 2.7. *Let $k \in \mathbb{N}$. Let D_A be an adjacency digraph that is an out-forest (similarly an in-forest) with n vertices and p connected components. Then $F(A^k) \geq F(A^{k+1})$, and so the sequence $\{F(A^k)\}_{k=1}^{\infty}$ is monotonically decreasing*

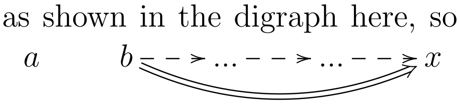
Proof. Let $n, p, k \in \mathbb{N}$. Let D_A be an adjacency digraph that is an out-forest with n vertices and p connected components. From Corollary 2.6, we know that D_{A^k} and $D_{A^{k+1}}$ are also out-forests. We have that every vertex has an indegree of 1 or 0 in D_{A^k} and $D_{A^{k+1}}$. Consider an arbitrary vertex x in $D_{A^{k+1}}$. We examine the following cases regarding x:

(Case 1) x has an indegree of 1 in $D_{A^{k+1}}$.

(Case 1a) x has an indegree of 0 in D_{A^k}. Since x has an indegree of 1 in $D_{A^{k+1}}$, then there exists a path of length $k+1$ from some vertex a to x and another path of length k from some vertex b to x, so this case is impossible. (Case 1b) x has an indegree of 1 in D_{A^k}. By the logic of Case1a, this case is possible.

(Case 2) x has an indegree of 0 in $D_{A^{k+1}}$.

(Case 2a) x has an indegree of 0 in D_{A^k}. This case is possible trivially. (Case 2b) x has an indegree of 1 in D_{A^k}. Since x has an indegree of 0 in $D_{A^{k+1}}$, there exists no path of length $k+1$ from some vertex a to x. However, there can exist a path of length k from some vertex b to x, as shown in the digraph here, so this case is possible. D_A (normal arrows), D_{A^k} (thicker arrow)



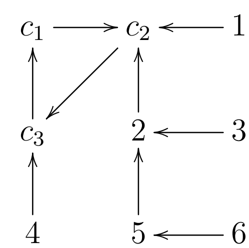
Since D_{A^k} and $D_{A^{k+1}}$ are out-forests, we need not consider vertices having an indegree higher than 1. We have that every vertex with an indegree of 1 in $D_{A^{k+1}}$ will also have exactly an indegree of 1 in D_{A^k}. Note that the number of vertices with an indegree of 1 in $D_{A^{k+1}}$ is exactly $F(A^{k+1})$. We also note that an arbitrary vertex in $D_{A^{k+1}}$ with an indegree of 0 can have an indegree of 0 or 1 in D_{A^k}. We denote the number of vertices which have an indegree of 0 in $D_{A^{k+1}}$ but have an indegree of 1 in D_{A^k} as $q \in \mathbb{N}_0$. Then $F(A^{k+1}) \leq F(A^{k+1}) + q = F(A^k)$. Thus, $F(A^k) \geq F(A^{k+1})$, and so the sequence $\{F(A^k)\}_{k=1}^{\infty}$ is monotonically decreasing. Observe that $D_{(A^T)^k}$ corresponds to an arbitrary in-forest. By Theorem 2.1, we have that for any k, $F(A^k) = F((A^T)^k)$, and so the sequence $\{F((A^T)^k)\}_{k=1}^{\infty}$ is also monotonically decreasing, as desired. □

Below is a theorem that shows the monotonicity of cycles.

Theorem 2.8. *If D_A is an adjacency digraph that is a cycle of length $n \in \mathbb{N}$, then $\{F(A^k)\}_{k=1}^{\infty} = n$*

Proof. This is obvious, no need for a proof. □

In the theorems below we will show that a digraph that is a cycle of arbitrary size with the vertices of the cycle being the roots of out-trees or in-trees is monotonic. That is, there are directed rooted trees whose roots are "planted" in the cycle. The trees connected to the cycle must be of the same kind, specifically all in-trees or all out-trees. A digraph example is shown to the right. Note that the total number of vertices in the two distinct in-trees is 6 (not including the cycle's vertices), with c_2 being the root of one of the in-trees, and with c_3 being the root of the other in-tree.



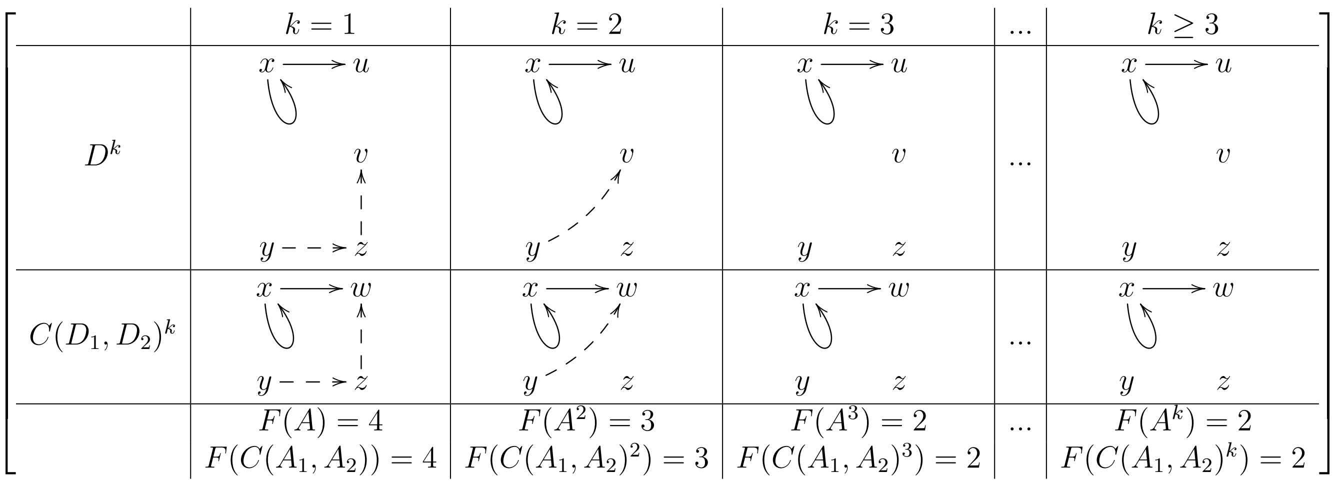
Theorem 2.9. *Let D_A be an adjacency digraph containing a cycle with the vertices of the cycle being the roots of in-trees (similarly out-trees). Let $k \in \mathbb{N}$. Then the sequence $\{F(A^k)\}_{k=1}^{\infty}$ is constant.*

Proof. Let $k, p \in \mathbb{N}$ and $m_i \in \mathbb{N}_0$. Suppose we have a cycle with p vertices. Label the vertices of the cycle as $\{c_1, c_2, ..., c_p\}$. Now suppose at each c_i, we let there exist an in-tree (similarly an out-tree) connected to the cycle with c_i being the root vertex. Denote the number of vertices in each in-tree (similarly out-tree) as m_i, where the root vertex c_i is not counted in m_i. Denote the sum of all distinct in-tree (similarly out-tree) vertices to be $m = m_1 + m_2 + ... + m_p$. Let D_A be an adjacency digraph containing a cycle with the vertices of the cycle being the roots of in-trees as described above. In D_A, note that every vertex has an outdegree of exactly 1. Then starting at any vertex in D_A, there is a unique path of length k leading to another vertex in D_A. This means that in D_{A^k}, every vertex has an outdegree of exactly 1. Since there are $p + m$ total vertices, we have $F(A^k) = p + m$, and so the sequence $\{F(A^k)\}_{k=1}^{\infty}$ is constant. Observe that D_{A^T} is an adjacency digraph that is a cycle with the vertices of the cycle being the roots of out-trees as described earlier in the proof. By Theorem 2.1, we have that for any k, $F(A^k) = F((A^T)^k)$, and so the sequence $\{F((A^T)^k)\}_{k=1}^{\infty} = p + m$, as desired. □

The following definition refers to the process of vertex identification (which is the same as edge contraction without needing an edge between two vertices, see [6] for more information). By using vertex identification, we can determine the number of edges in a larger digraph by considering another similar digraph whose components are the similar digraph's subgraphs, whose number of edges we know to be monotonic from previous research.

Definition 2.10. *Let $k \in \mathbb{N}$. In a digraph D^k, vertex identification is the replacement of two distinct vertices u and v (within distinct components) with a single vertex w such that the edges incident to w are the edges that were incident with u and v. For brevity, we can denote the subgraphs corresponding to the distinct components of D^k as D_1^k and D_2^k, calling them component subgraphs of D^k, such that $D_1^k \cup D_2^k = D^k$. We denote the resulting graph after vertex identification (remembering to remove any duplicate loops) as G. If u and v are both sources (or sinks), we call $G = C(D_1^k, D_2^k)$ the compound digraph of D^k, and if both vertices also have a loop, we call $G = C_L(D_1^k, D_2^k)$ the compound loop digraph of D^k. In this specific case, we refer to D_1^k and D_2^k as component loop subgraphs of D^k. The vertices u and v must both have a loop or both not have a loop in D_1^k and D_2^k.*

An example that uses the definition above is seen in the chart below. Here we have a digraph D^k with vertices u and v (both sinks) in distinct components, that can be "combined" using vertex identification into the compound digraph $C(D_1, D_2)$, where D_1, D_2 are component subgraphs of D. For the example below, let A^k and $C(A_1, A_2)^k$ be the adjacency matrices corresponding to D^k and $C(D_1, D_2)^k$ respectively.



This example gleans to us some intuition on how we should use the definition of compound digraphs. For instance, suppose we have a digraph $C(D_1, D_2)$ that has a sink or a source at some vertex w. Now computing $C(D_1, D_2)^2$, $C(D_1, D_2)^3$,... may be too cumbersome to do with a larger digraph. This makes it difficult to determine the monotonicity of the adjacency matrix corresponding to $C(D_1, D_2)$. So, what if we instead considered a digraph D with two distinct components that when "combined" create the compound digraph $C(D_1, D_2)$. It likely will be easier to determine the monotonicity of the adjacency matrices corresponding to the component subgraphs D_1 and D_2, especially since in [2] and [9] we have that digraphs with 4 or less edges maintain monotonicity in almost all cases. Letting A_1^k, A_2^k, and $C(A_1, A_2)^k$ be the adjacency matrices corresponding to D_1^k, D_2^k, and $C(D_1, D_2)^k$ respectively, if we could show $F(A_1^k) + F(A_1^k) = F(C(A_1, A_2)^k)$, then when presented with a compound digraph with a source or a sink, we could simply examine the component subgraphs of the similar digraph D to determine monotonicity. The following theorems together prove this.

We let A^k, A_1^k, A_2^k, $C(A_1, A_2)^k$, and $C(A_1^k, A_2^k)$ be the adjacency matrices corresponding to the digraphs D^k, D_1^k, D_2^k, $C(D_1, D_2)^k$, and $C(D_1^k, D_2^k)$ respectively for the following theorems.

Theorem 2.11. *Let $k \in \mathbb{N}$. Suppose we have an adjacency digraph D^k with two component adjacency subgraphs D_1^k and D_2^k. Then $F(A^k) = F(A_1^k) + F(A_2^k) = F(C(A_1^k, A_2^k))$.*

Proof. This is obvious, no need for a proof. □

Theorem 2.12. *Let $k \in \mathbb{N}$. Suppose we have a digraph D with two component subgraphs D_1 and D_2. Then $C(D_1^k, D_2^k) = C(D_1, D_2)^k$.*

Proof. Let $k \in \mathbb{N}$. Suppose we have a digraph D with two component subgraphs D_1 and D_2. Let $D_1 = (V_1, E_1)$ and $D_2 = (V_2, E_2)$. Let $u \in V_1$ and $v \in V_2$ be vertices which are both sinks or both sources. Consider the vertex sets and edge sets of $C(D_1^k, D_2^k)$ and $C(D_1, D_2)^k$. The vertex sets of $C(D_1^k, D_2^k)$ and $C(D_1, D_2)^k$ are equal trivially.

(Proving the edge sets of $C(D_1^k, D_2^k)$ and $C(D_1, D_2)^k$ are equal) [*Edge set of* $C(A_1^k, A_2^k)$] By the definition of the component subgraphs D_1 and D_2. we have that their edge sets E_1 and E_2 are disjoint. Then there is no path of any length leading from some vertex in V_1 to another vertex in V_2. Take D_1 and D_2 to the kth power. Let E_1' and E_2' denote the edges sets of D_1^k and D_2^k respectively. We find the compound digraph of D^k, namely $C(D_1^k, D_2^k)$, letting w be the vertex in $C(D_1^k, D_2^k)$ which replaced u and v. The edge set of $C(D_1^k, D_2^k)$, denote it as E', is the union of E_1' and E_2', but for all ordered pairs in E_1' and E_2' that contain u and v respectively, we replace u and v with w. [*Edge set of* $C(D_1, D_2)^k$] Taking our component subgraphs D_1 and D_2, we find the compound digraph of D, namely $C(D_1, D_2)$, letting w be the vertex in $C(D_1, D_2)$ which replaced u and v. We specified that u and v were both sources or were both sinks, so the vertex w must be a source or a sink. Since w is a source or a sink in $C(D_1, D_2)$, it is impossible for a path of length k to go from some vertex in $V_1 \setminus \{u\}$ to a vertex in $V_2 \setminus \{v\}$, and vice versa. Then we have that the only paths of length k from some vertex r in $C(D_1, D_2)$ to some vertex s in $C(D_1, D_2)$ are contained within the subset $(V_1 \setminus \{u\}) \cup \{w\}$ or within the subset $(V_2 \setminus \{v\}) \cup \{w\}$. Examining this statement, it correlates to the union of the edge sets E_1' and E_2' as defined earlier. So, the edge set of $C(D_1, D_2)^k$ is E'.

Thus we have shown that the vertex and edge sets of $C(D_1^k, D_2^k)$ and $C(D_1, D_2)^k$ are equal. Hence $C(D_1^k, D_2^k) = C(D_1, D_2)^k$, as desired. □

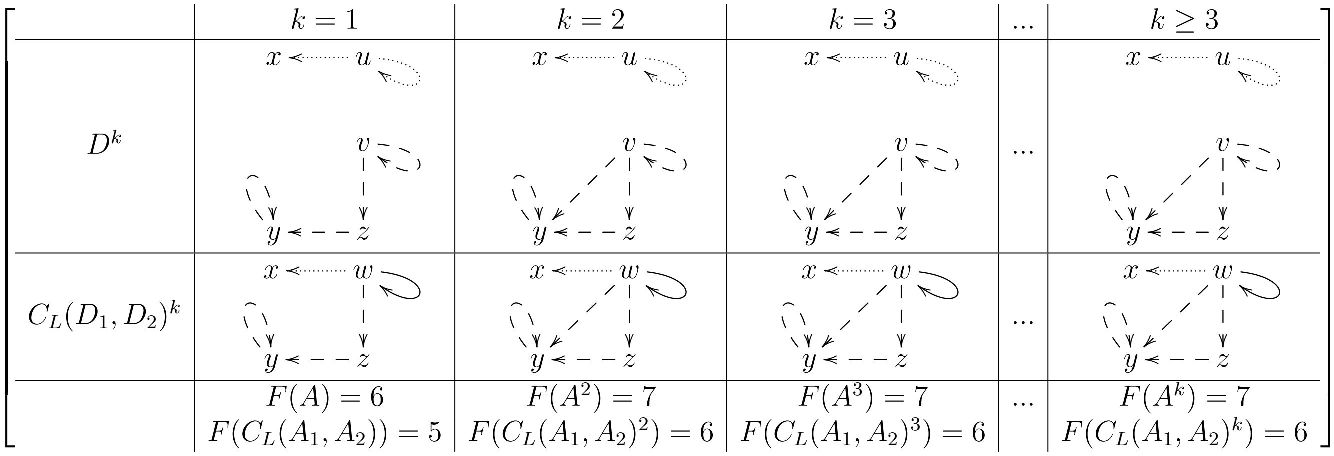
Corollary 2.13. *Let $k \in \mathbb{N}$. Suppose we have an adjacency digraph D with two component adjacency subgraphs D_1 and D_2. Then $F(A^k) = F(A_1^k) + F(A_2^k) = F(C(A_1, A_2)^k)$.*

Proof. Follows directly from combining Theorems 2.11 and 2.12. □

Corollary 2.14. *Let $k \in \mathbb{N}$. Suppose we have an adjacency digraph D with two component adjacency subgraphs D_1 and D_2. If the sequences $\{F(A_1^k)\}_{k=1}^{\infty}$ and $\{F(A_2^k)\}_{k=1}^{\infty}$ are both monotonically decreasing (similarly both monotonically increasing), then the sequence $\{F(C(A_1, A_2)^k)\}_{k=1}^{\infty}$ will be monotonically decreasing (similarly monotonically increasing).*

Proof. Let $k \in \mathbb{N}$. Suppose we have an adjacency digraph D with two component adjacency subgraphs D_1 and D_2, and the sequences $\{F(A_1^k)\}_{k=1}^{\infty}$ and $\{F(A_2^k)\}_{k=1}^{\infty}$ are both monotonically decreasing. Then $F(A_1^k) \geq F(A_1^{k+1})$ and $F(A_2^k) \geq F(A_2^{k+1})$. Applying Corollary 2.13 twice, we have $F(C(A_1, A_2)^k) = F(A_1^k) + F(A_2^k) \geq F(A_1^{k+1}) + F(A_2^{k+1}) = F(C(A_1, A_2)^{k+1})$, as desired. Similarly, we have $F(C(A_1, A_2)^k) = F(A_1^k) + F(A_2^k) \leq F(A_1^{k+1}) + F(A_2^{k+1}) = F(C(A_1, A_2)^{k+1})$ if the sequences $\{F(A_1^k)\}_{k=1}^{\infty}$ and $\{F(A_2^k)\}_{k=1}^{\infty}$ are both monotonically increasing, as desired. □

Now, we consider an example that uses the definition of a compound loop digraph. In the chart below we have a digraph D with two component loop subgraphs, D_1 and D_2, that can be "combined" using vertex identification into the compound loop digraph $C_L(D_1, D_2)$ of D. For the example below, let A^k and $C_L(A_1, A_2)^k$ be the adjacency matrices corresponding to D^k and $C_L(D_1, D_2)^k$ respectively.



Similar to compound digraphs, utilizing compound loop digraphs allows us to more easily determine the monotonicity of a compound loop digraph $C_L(D_1, D_2)$. The one key difference is that when the component loop subgraphs are "combined," the two loops become one in the compound loop digraph, causing there to be a difference of exactly one edge. We let A_1^k, A_2^k, and $C_L(A_1, A_2)^k$ be the adjacency matrices corresponding to the digraphs D_1^k, D_2^k, and $C_L(D_1, D_2)^k$ respectively for the following theorem.

Theorem 2.15. *Let $k \in \mathbb{N}$. Suppose we have an adjacency digraph D with two component loop adjacency subgraphs D_1 and D_2. If the sequences $\{F(A_1^k)\}_{k=1}^{\infty}$ and $\{F(A_2^k)\}_{k=1}^{\infty}$ are both monotonically decreasing (similarly both monotonically increasing), then the sequence $\{F(C_L(A_1, A_2)^k)\}_{k=1}^{\infty}$ will be monotonically decreasing (similarly monotonically increasing).*

Proof. (Since the proof leading to this theorem is so similar to the culmination of proofs from Theorems/Corollaries 2.11 - 2.14, we do not write out the full proof here.) □

3 Classification of cases

The theorems proved above allow us to not have to consider certain digraph cases. Presented below are theorems found in [2] that will help us prove monotonicity for our cases:

Definition 3.1. *Let $k, m \in \mathbb{N}$. We say that a zero-one matrix A is k-periodic starting at m if $A^m = A^{m+k}$.*

Theorem 3.2. *Let the zero-one matrix A be k-periodic starting at m for some $k, m \in \mathbb{N}$ with $F(A^m) = F(A^{m+1}) = \cdots = F(A^{m+k-1})$. Then $\{F(A^n)\}_{n=m}^{\infty}$ is constant.*

Definition 3.3. *Let $k > 0$. We say that a zero-one matrix A is k-stable if A is 1-periodic starting at k.*

Corollary 3.4. *Let A be a k-stable zero-one matrix. Then $\{F(A^n)\}_{n=k}^{\infty}$ is constant.*

We will now create a key that will describe why we found the number of edges in each digraph case to be monotonic. The cases that are already shown to be monotonic due to the theorems listed above will not be included below for the sake of brevity. Below are our classifications of cases: we found (through direct computation) that the number of edges for a particular adjacency digraph was monotonic because its corresponding adjacency matrix A eventually...

I is k-periodic starting at m for some $k, m \in \mathbb{N}$ with $F(A^m) = F(A^{m+1}) = ... = F(A^{m+k+1})$, so we have that the sequence $\{F(A^n)\}_{n=m}^{\infty}$ is constant by Theorem 3.2. We also found for the digraph that $\{F(A^n)\}_{n=1}^{(m-1)}$ is monotonic. Combining, we have that $\{F(A^n)\}_{n=1}^{\infty}$ is monotonic.

II is k-stable, with $k \in \mathbb{N}$. Then we have that the sequence $\{F(A^n)\}_{n=k}^{\infty}$ is constant by Corollary 3.4. We also found for the digraph that $\{F(A^n)\}_{n=1}^{(k-1)}$ is monotonic. Combining, we have that $\{F(A^n)\}_{n=1}^{\infty}$ is monotonic.

III becomes the zero-matrix at some A^k, with $k \in \mathbb{N}$. We also found for the digraph that the sequence $\{F(A^n)\}_{n=1}^{k}$ is monotonic decreasing. Thus, we have $\{F(A^n)\}_{n=1}^{\infty}$ is monotonically decreasing.

We now must find all of the adjacency digraph cases with exactly 5 directed edges. Below is our methodology for finding such cases:

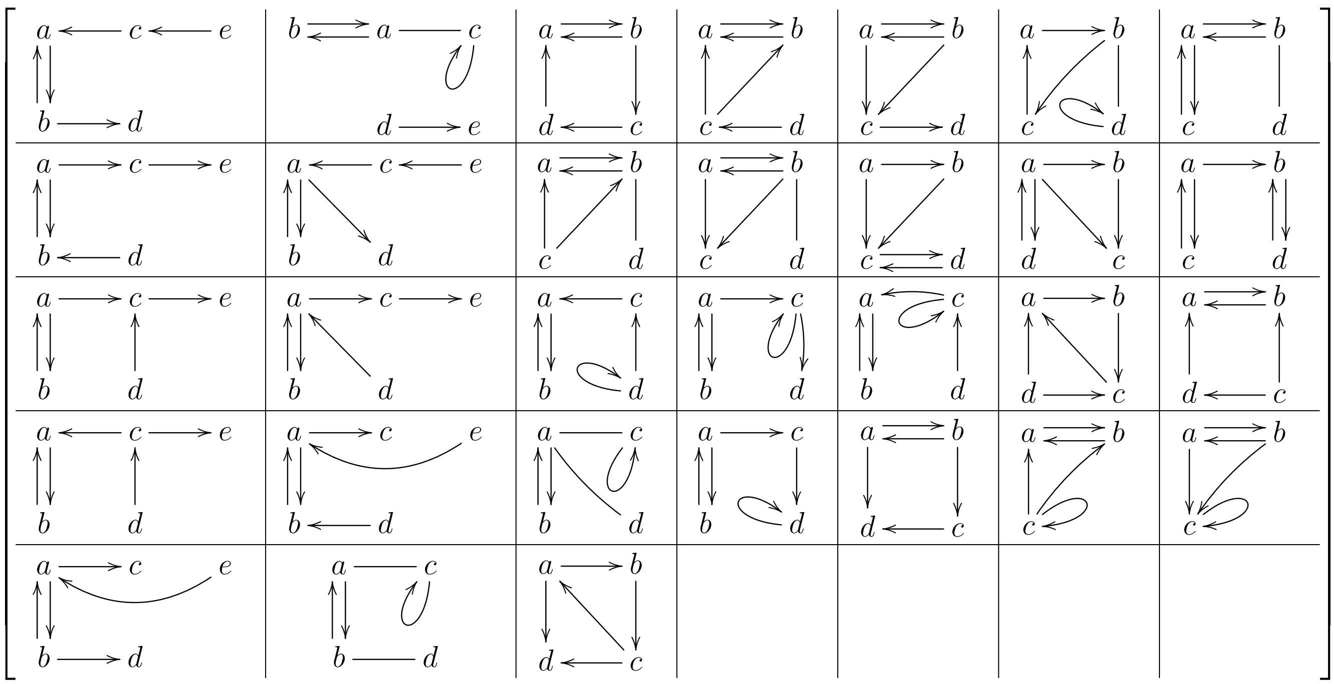
Step 1: Let's assume our graphs: (1) are undirected, (2) have at most 2 edges between any two vertices, (3) have no loops, (4) have every vertex having degree at least 2, and (5) have at most 5 edges.
Step 2: List all graphs that fall into the above category. Note one of these graphs will have no vertices (call this graph X).
Step 3: For graph X, we list graphs with an undirected tree with (a) 5 edges, (b) 4 edges, (c) 3 edges, (d) 2 edges, (e) and 1 edge. We also list graphs with (f) no undirected tree. Once we have these graphs, add disjoint undirected trees (that do not share vertices with the existing tree in each graph) to have at most 5 edges. List these graphs. Now skip to Step 5, performing Step 5 for all these graphs resulting from the original graph X.
Step 4: Once we have the graphs from Step 4 (not including graph X), add undirected trees (either to existing vertices or by creating disjoint undirected trees), to have at most 5 edges. List these graphs.
Step 5: Once we have those, we add enough loops (either to existing vertices or creating new disjoint vertices) to get up to 5 edges. List these graphs.
Step 6: Once we have all these graphs, make them directed and test all varying indegree and outdegree for vertices.

The above methodology lists all the digraph cases with 5 directed edges. In the cases shown below, we do not include cases we know maintain monotonicity from our theorems in Section 2.

3.1 Cases exhibiting k-periodicity for some A^k (Classification I)

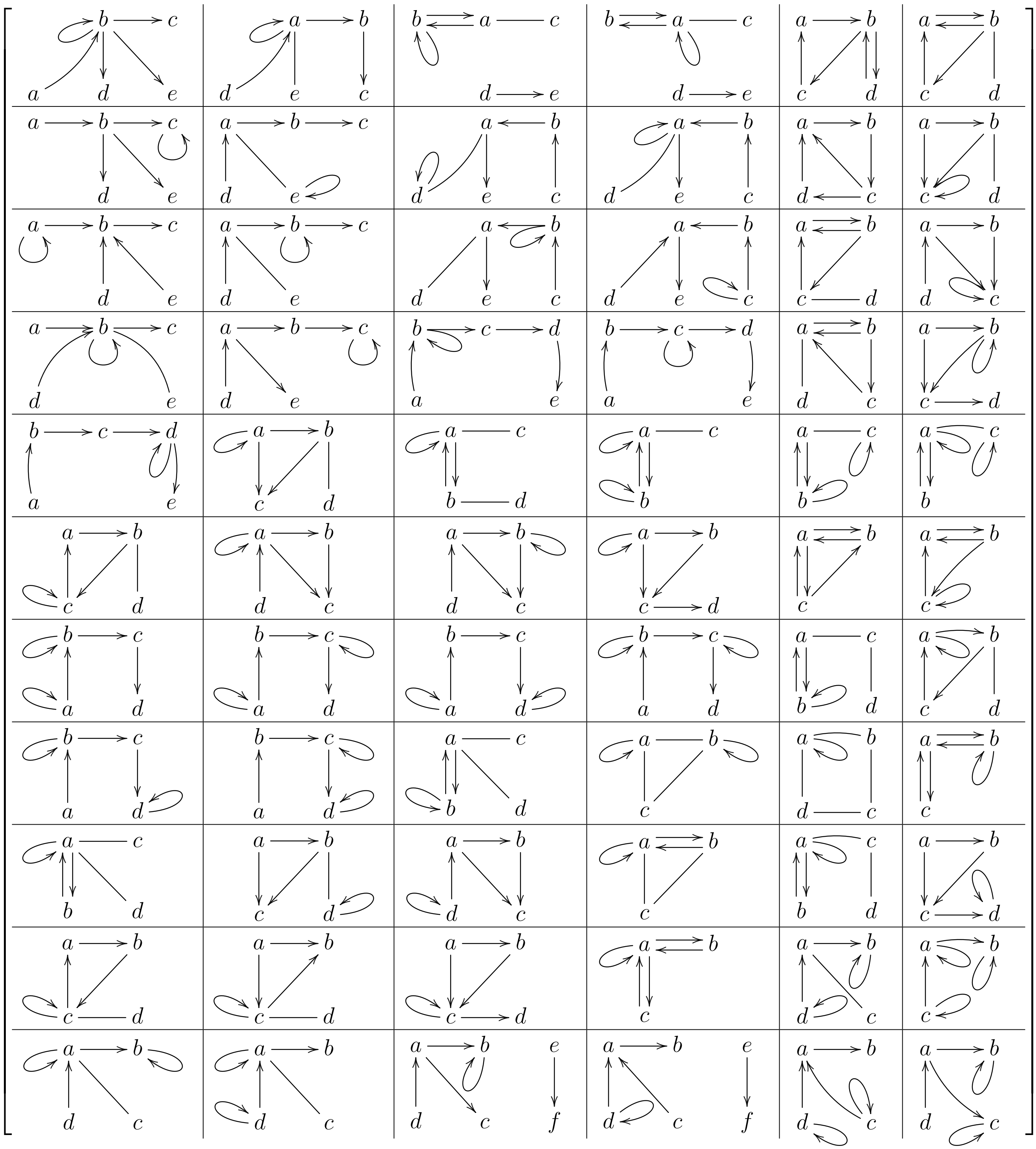
All the adjacency digraphs below meet the requirements of Classification I. For instance, the adjacency digraph D_A at the end of the third row has (through direct computation) $A^2 = A^4$, $A^3 = A^5$,

$F(A) = 5$, $F(A^2) = 4$, and $F(A^3) = 4$. From Theorem 3.2, the sequence $\{F(A^n)\}_{n=1}^{\infty}$ is monotonically decreasing.



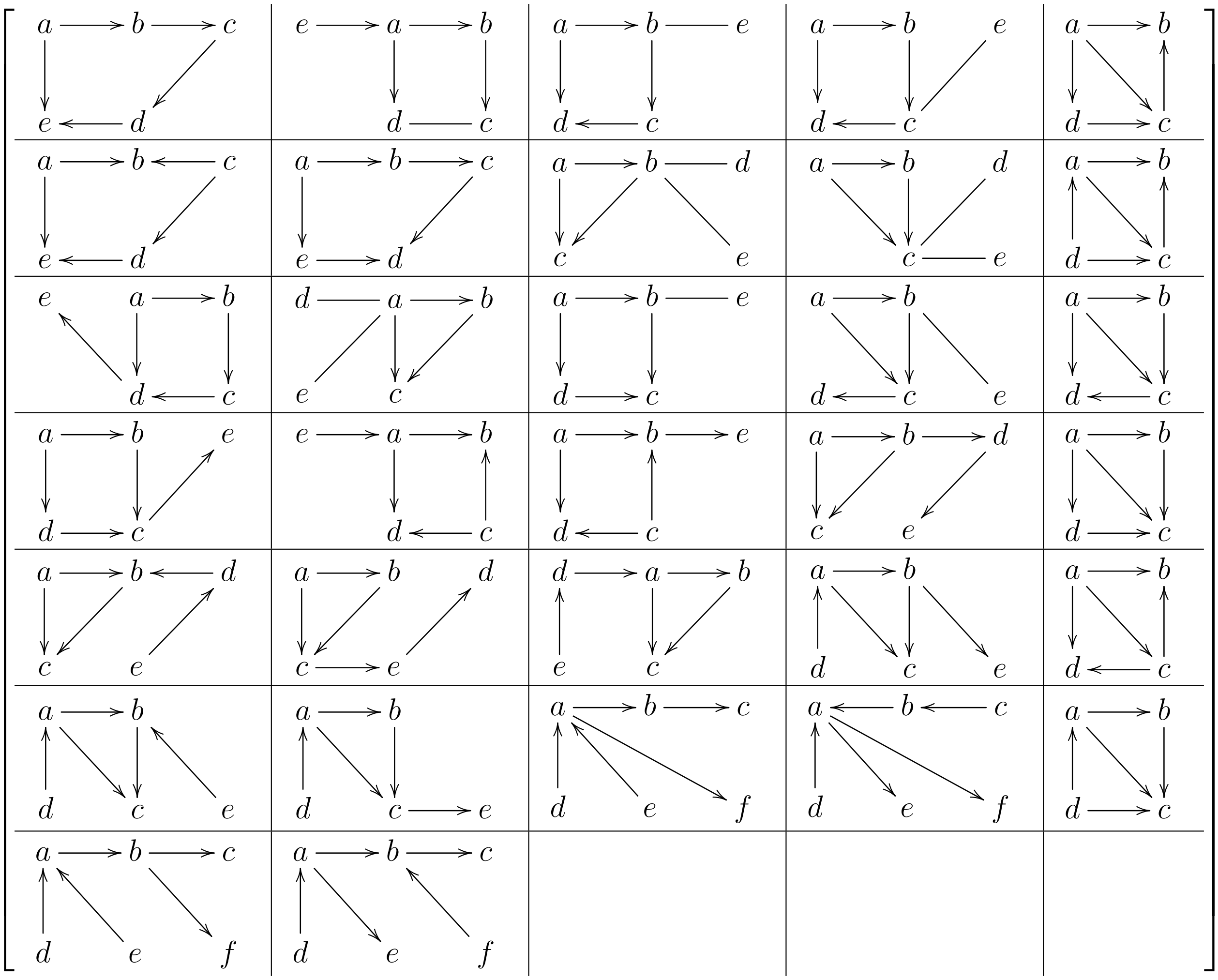
3.2 Cases becoming k-stable for some A^k (Classification II)

All the adjacency digraphs below meet the requirements of Classification II. For instance, the adjacency digraph D_A at the fifth entry of the first row has A becoming (through direct computation) 6-stable. Also, the sequence $\{F(A^n)\}_{n=1}^{6}$ is monotonic increasing. From Corollary 3.4, the sequence $\{F(A^n)\}_{n=1}^{\infty}$ is monotonically increasing.



3.3 Cases becoming the zero matrix at some A^k (Classification III)

All the adjacency digraphs below meet the requirements of Classification III. For instance, the adjacency digraph D_A at the first entry of the first row has (through direct computation) A^5 becoming the zero-matrix. Also, the sequence $\{F(A^n)\}_{n=1}^{5}$ is monotonically decreasing. From Corollary 3.4, the sequence $\{F(A^n)\}_{n=1}^{\infty}$ is monotonically decreasing.

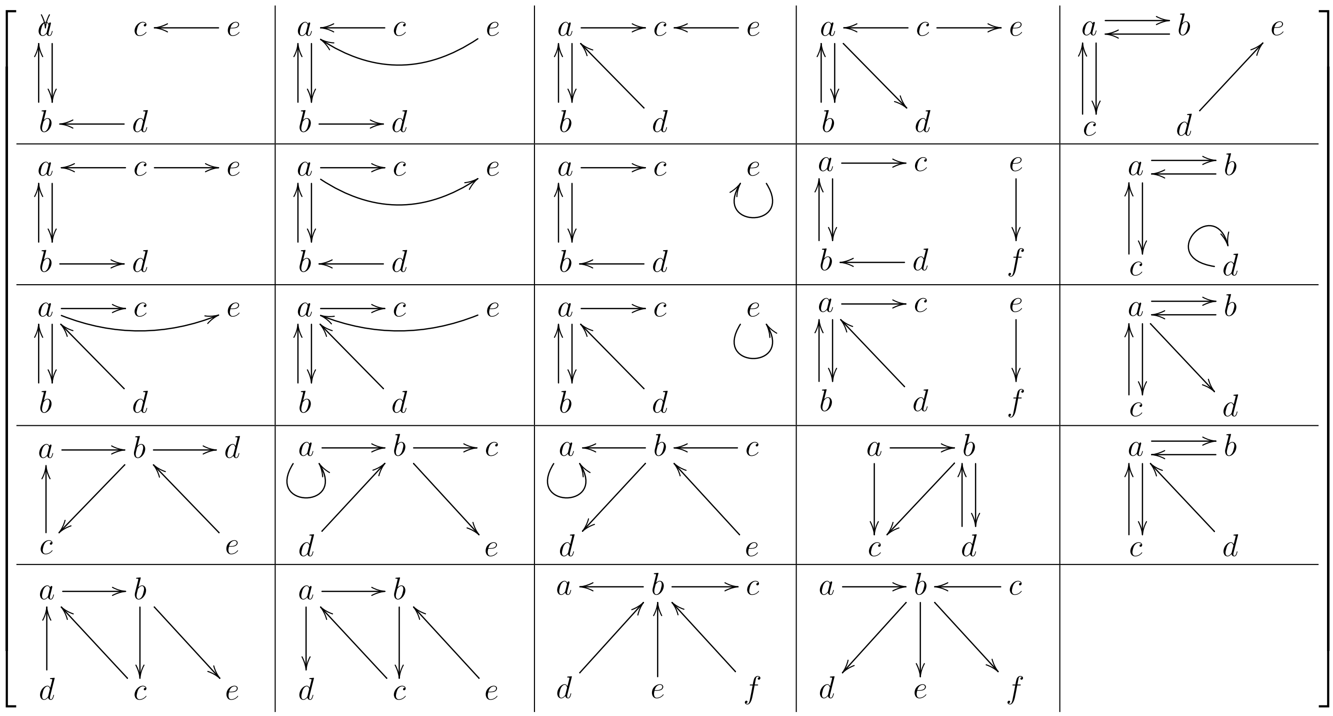


Cases that were not listed above have already been proved to be monotonic due to past papers and our theorems.

4 Non-monotonic Cases / Conclusion

Through direct computation, we found that the digraph cases listed below corresponding to adjacency matrices have the sequence $\{F(A^n)\}_{n=1}^{\infty}$ being non-monotonic. For instance, the adjacency digraph D_A at the last entry of the third row has $A = A^3$, $A^2 = A^4$, $F(A) = 5$, and $F(A^2) = 7$. Then A is 2-periodic starting at 1 and 2. Then, we have $\{F(A^{2k+1})\}_{k=0}^{\infty} = 5 \neq 7 = \{F(A^{2k})\}_{k=1}^{\infty}$. Hence, we have the sequence $\{F(A^k)\}_{k=1}^{\infty}$ is non-monotonic as the number of positive entries in our adjacency matrix oscillates between 5 and 7.

Theorem 4.1. *Let $k \in \mathbb{N}$. Suppose we have a square 0-1 matrix A with $F(A) = 5$. Then the sequence $\{F(A^k)\}_{k=1}^{\infty}$ is monotonic unless D_A is one of the following digraphs:*



Proof. Assume A is also an adjacency matrix for a digraph D_A. We prove by cases:
(Case 1) Assume D_A is an out-forest or an in-forest with an arbitrary number of connected components. Then by Theorem 2.7 we have that $F(A^k) \geq F(A^{k+1})$, meaning the sequence is monotonically decreasing.
(Case 2) Assume D_A is a cycle of length 5. Then by Theorem 2.8, the sequence $\{F(A^k)\}_{k=1}^{\infty} = 5$.
(Case 3) Assume D_A is a cycle with the vertices of the cycle being the roots of in-trees (similarly out-trees) as described in the proof of Theorem 2.9. Then by Theorem 2.9, we have the sequence $\{F(A^k)\}_{k=1}^{\infty}$ is constant.

The next two cases rely primarily on how we know the monotonicity of matrices corresponding to digraphs with 4 edges or less are almost all monotonic.
(Case 4) Assume D_A is a compound digraph $C(D_1, D_2)$ for a digraph D that can be formed by two component subgraphs, D_1 and D_2, of D. We let A_1^k, A_2^k, and $C(A_1, A_2)^k$ be the adjacency matrices corresponding to the digraphs D_1^k, D_2^k, and $C(D_1, D_2)^k$ respectively. Then by Corollary 2.14, if we know the sequences $\{F(A_1^k)\}_{k=1}^{\infty}$ and $\{F(A_2^k)\}_{k=1}^{\infty}$ are both monotonically decreasing (similarly both monotonically increasing), then the sequence $\{F(C(A_1, A_2)^k)\}_{k=1}^{\infty}$ will be monotonically decreasing (similarly monotonically increasing).
(Case 5) Assume D_A is a compound loop digraph $C_L(D_1, D_2)$ for a digraph D that can be formed by two component loop subgraphs, D_1 and D_2, of D. We let A_1^k, A_2^k, and $C_L(A_1, A_2)^k$ be the adjacency matrices corresponding to the digraphs D_1^k, D_2^k, and $C_L(D_1, D_2)^k$ respectively. Then by Theorem 2.15, if we know the sequences $\{F(A_1^k)\}_{k=1}^{\infty}$ and $\{F(A_2^k)\}_{k=1}^{\infty}$ are both monotonically decreasing (similarly both monotonically increasing), then the sequence $\{F(C_L(A_1, A_2)^k)\}_{k=1}^{\infty}$ will be monotonically decreasing (similarly monotonically increasing).
(Case 6) For all other cases that are not covered by the previous cases, we prove with side calculations. These cases and the methodology for finding them are listed in Section 3 of this paper.

In all cases, we get the desired result. □

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