Sums of Generalized Octahedral Numbers

Over 150 years ago, Pollock conjectured that every positive integer may be expressed as the sum of at most five tetrahedral numbers, and also as the sum of at most seven octahederal numbers. These figurate numbers $Te_n = \frac{(n+2)(n+1)n}{6}$ and $O_n = \frac{n(2n^2+1)}{3}$, for $n \geq 0$, represent the quantity of close-packed spheres in a tetrahedral or octahedral shape, with side length n.

Recently, it was proven (in [**1**]) that every integer greater than e^{10^7} is the sum of seven octahedral numbers. Also, attention has been given (in [**3, 4**]) to a generalized version of Pollock's tetrahedral conjecture, by considering generalized Te_n (i.e. where n is allowed to be any integer). We now offer our final installment of these miniatures: on the generalized version of Pollock's octahedral conjecture. Five generalized O_n suffice to represent all integers, but two do not.

Theorem. *Every integer is the sum of five generalized octahedral numbers. However, if p is prime and $p \equiv 3 \pmod 7$, then p is not the sum of two generalized octahedral numbers.*

Proof. For the first statement, we use a common "forward difference" trick (see, e.g., [**3**], or problem 227 in [**2**]). We observe that for every integer m, we have $4m = O_{m+1} + O_{m-1} + O_{-m} + O_{-m} = \frac{1}{3}((m+1)(2(m+1)^2+1) + (m-1)(2(m-1)^2+1) + 2(-m)(2(-m)^2+1))$. Next, we observe that $O_0 = 0, O_1 = 1, O_2 = 6, O_3 = 19$, and these are all different modulo 4. Now, for any integer n, we see that $n - O_i$ is a multiple of 4 (and hence representable as the sum of four generalized octahedral numbers), for some $i \in \{0, 1, 2, 3\}$.

For the second statement, we mimic the proofs in [**4**]. Let p be prime and either $p = O_n + O_{n-k}$ or $p = O_n + O_{-(n-k)}$ for $n \geq k \geq 0$. Supposing the first case, $3p = (2n-k)(2k^2 - 2kn + 2n^2 + 1)$. If $(2n-k)|3$, this gives the small set of (n, k) pairs $\{(3,3), (2,1), (1,1)\}$. If instead $(2k^2 - 2kn + 2n^2 + 1)|3$, this forces $n < 2$, which gives the (n, k) pair $\{(1, 0)\}$. These cases correspond to primes $p = 2, 7, 19$ only. Supposing the second case, $3p = k(2k^2 - 6kn + 6n^2 + 1)$. Set $t = 2k^2 - 6kn + 6n^2 + 1$. If $k = 3$ then t is equivalent to one of $\{0, 1, 2, 5\}$, modulo 7; if $k = 1$ then $\frac{t}{3}$ is equivalent to one of $\{1, 4, 5, 6\}$, modulo 7. Since $p \equiv 3 \pmod 7$, none of these are possible. ∎

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