Primes in (0, *n*] vs. (*n*, *m*]

The prime number theorem gives us an approximate distribution of primes, but what can be said about the relative number of primes in the intervals $(0, n]$ and $(n, m]$, for natural numbers $m > n$? The case $m = 2n \gg 0$ was known clasically, but recent advances allow for a sharper result.

Suppose that $m > n \geq 5393$. These nice bounds for the prime counting function π were proved in [**1**]:

$$\frac{n}{\log n - 1} < \pi(n) \leq \pi(m) < \frac{m}{\log m - 1.112}.$$

Theorem. *Suppose that* $m > n \geq 5393$. *Suppose further that* $\frac{m}{n} > e^{0.112} \approx 1.12$. *Then*

$$\frac{\pi(n)}{n} = \frac{\pi(n) - \pi(0)}{n - 0} > \frac{\pi(m) - \pi(n)}{m - n}.$$

Proof. $\frac{m}{n}\pi(n) - \pi(m) > \frac{m}{n}\frac{n}{\log n-1} - \frac{m}{\log m-1.112} = m\left(\frac{1}{\log n-1} - \frac{1}{\log m-1.112}\right)$. This is positive, since $\frac{m}{n} > e^{0.112}$ and thus $\log n - 1 < \log m - 1.112$. We now rearrange $\frac{m}{n}\pi(n) - \pi(m) > 0$ into the desired statement. ■

REFERENCES

1. Dusart, P. (2018). Explicit estimates of some functions over primes. *Ramanujan J.* 45(1): 227–251. `doi.org/10.1007/s11139-016-9839-4`

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