Gelfand’s Question in Different Bases

Vadim Ponomarenko

Department of Mathematics and Statistics
San Diego State University

Joint Math Meetings  January 4, 2017

http://www-rohan.sdsu.edu/~vadim/gelfand-talk.pdf
Please encourage your students to apply to the
San Diego State University Mathematics REU.
Summer 2017 projects: number theory, hydrodynamics

http://www.sci.sdsu.edu/math-reu/index.html

This work was done jointly with Jason Thoma, Master’s student.
Let $\langle n \rangle$ denote the leading digit of positive integer $n$.
e.g. $\langle 12 \rangle = 1$, $\langle 345 \rangle = 3$, $\langle 7 \rangle = 7$

Question 1: (Gelfand? 1965?)  
Is there any $n \in \mathbb{N}$ with $\langle 2^n \rangle = 9$?

Question 2: Set $D = \{1, 2, \ldots, 9\}$, the nonzero digits.  
Given $d, t \in D$, is there any $n \in \mathbb{N}$ with $\langle d^n \rangle = t$?

Note: $d = 1$ is trivial, as $d^n = 1$. 
More Background

Question 3:

Given a vector \( t \), i.e. \( t \in D^8 \), is there any \( n \in \mathbb{N} \) with \( t \) achieved, i.e. \( (\langle 2^n \rangle, \langle 3^n \rangle, \ldots, \langle 9^n \rangle) = t \)？

Special cases:

\( t = (2, 3, \ldots, 9) \), insisting that \( n > 1 \)
\( t = (a, a, \ldots, a) \), for some \( a \in D \)
\( t_1 t_2 \cdots t_8 \), viewed as an 8-digit number, is prime
Eising, Radcliffe, Top paper

American Mathematical Monthly 122 (3) 2015
Eising, Radcliffe, Top
“A Simple Answer to Gelfand’s Question”

Q1: $\langle \langle 2^n \rangle \rangle = 9$? Yes
Q2: $d, t \in D, \langle \langle d^n \rangle \rangle = t$? Yes
Q3: $t \in D^8$, $t$ achieved?
17596 vectors $t$ are achieved (out of $9^8 = 43046721$)
23456789 is not achieved, nor is any $aaaaaaaaaa$
1127 primes are achieved
Kronecker’s Theorem (1884):

Let $x_1, x_2, \ldots, x_k \in \mathbb{R}$. Set $y$ to be the natural projection of $(x_1, x_2, \ldots, x_k)$ into the additive group $\mathbb{R}^k / \mathbb{Z}^k$. TFAE:

(1) \{1, x_1, \ldots, x_k\} is $\mathbb{Q}$-linearly independent

(2) $\langle y \rangle$ is dense in $\mathbb{R}^k / \mathbb{Z}^k$. 
Kronecker’s Theorem, special case

Kronecker’s Theorem:
Let \( x_1, x_2, \ldots, x_k \in \mathbb{R} \). Set \( y \) to be the natural projection of \((x_1, x_2, \ldots, x_k)\) into the additive group \( \mathbb{R}^n/\mathbb{Z}^n \). TFAE:

1. \( \{1, x_1, \ldots, x_k\} \) is \( \mathbb{Q} \)-linearly independent
2. \( \langle y \rangle \) is dense in \( \mathbb{R}^k/\mathbb{Z}^k \).

Take \( k = 1 \). Then \( x \notin \mathbb{Q} \), if and only if \( \langle x \rangle \) is dense in \( \mathbb{R}/\mathbb{Z} \).
Kronecker’s Theorem, in ERT

Special Case: \( x \in \mathbb{R} \setminus \mathbb{Q} \), if and only if \( \langle x \rangle \) is dense in \( \mathbb{R}/\mathbb{Z} \).

Set \( \pi : \mathbb{R} \to \mathbb{R} \cap [0, 1) \) be the natural projection (mod 1).
ERT: \( \langle x \rangle = \lfloor 10^{\pi(\log_{10} x)} \rfloor \), where \( \lfloor \cdot \rfloor \) is the floor function.

Now set \( x = 2^n \). \( \langle x \rangle = \lfloor 10^{\pi(n \log_{10} 2)} \rfloor \).

Since \( \log_{10} 2 \notin \mathbb{Q} \), \( \langle \log_{10} 2 \rangle \) is dense in \( \mathbb{R}/\mathbb{Z} \). Hence for some \( n \), the exponent must be in \([\log_{10} 9, 1)\).
(Question #1) Note: \( n = 53 \) is smallest such \( n \).
Kronecker’s Theorem, in ERT

Special Case: $x \in \mathbb{R} \setminus \mathbb{Q}$, if and only if $\langle x \rangle$ is dense in $\mathbb{R}/\mathbb{Z}$.

Set $\pi : \mathbb{R} \to \mathbb{R} \cap [0, 1)$ be the natural projection (mod 1). ERT: $\langle x \rangle = \lfloor 10^{\pi(\log_{10} x)} \rfloor$, where $\lfloor \cdot \rfloor$ is the floor function.

Now set $x = 2^n$. $\langle x \rangle = \lfloor 10^{\pi(n \log_{10} 2)} \rfloor$.

Since $\log_{10} 2 \notin \mathbb{Q}$, $\langle \log_{10} 2 \rangle$ is dense in $\mathbb{R}/\mathbb{Z}$. Hence for some $n$, the exponent must be in $[\log_{10} 9, 1)$.

(Question #1) Note: $n = 53$ is smallest such $n$. 
Kronecker’s Theorem, in ERT

Special Case: \( x \in \mathbb{R} \setminus \mathbb{Q} \), if and only if \( \langle x \rangle \) is dense in \( \mathbb{R}/\mathbb{Z} \).

Set \( \pi : \mathbb{R} \rightarrow \mathbb{R} \cap [0, 1) \) be the natural projection (mod 1).
ERT: \( \langle \langle x \rangle \rangle = \lfloor 10^{\pi(\log_{10} x)} \rfloor \), where \( \lfloor \cdot \rfloor \) is the floor function.

Now set \( x = 2^n \). \( \langle \langle x \rangle \rangle = \lfloor 10^{\pi(n \log_{10} 2)} \rfloor \).

Since \( \log_{10} 2 \notin \mathbb{Q} \), \( \langle \log_{10} 2 \rangle \) is dense in \( \mathbb{R}/\mathbb{Z} \). Hence for some \( n \), the exponent must be in \( [\log_{10} 9, 1) \).
(Question #1) Note: \( n = 53 \) is smallest such \( n \).
What about other bases?

Let $B \in \mathbb{N}$ be our base, and $D = \{1, 2, \ldots, B - 1\}$.

Kronecker: $x \in \mathbb{R} \setminus \mathbb{Q}$, if and only if $\langle x \rangle$ is dense in $\mathbb{R}/\mathbb{Z}$.
ERT technique: $\langle \langle x \rangle \rangle = \lfloor B^\pi (\log_B x) \rfloor$.

If $B$ is not a perfect power, then $\log_B d \notin \mathbb{Q}$, and the same argument works; i.e. all $t$ are achieved.

If $B$ is a perfect power, then for certain $d$, $\log_B d \in \mathbb{Q}$, and $\langle \log_B d \rangle$ is not dense. What about $\langle \langle d^n \rangle \rangle$?

PT Thm: In that case certain $t$ are not achieved.
What about other bases?

Let $B \in \mathbb{N}$ be our base, and $D = \{1, 2, \ldots, B - 1\}$.

Kronecker: $x \in \mathbb{R} \setminus \mathbb{Q}$, if and only if $\langle x \rangle$ is dense in $\mathbb{R}/\mathbb{Z}$.
ERT technique: $\langle \langle x \rangle \rangle = \lfloor B^\pi (\log_B x) \rfloor$

If $B$ is not a perfect power, then $\log_B d \notin \mathbb{Q}$, and the same argument works; i.e. all $t$ are achieved.
If $B$ is a perfect power, then for certain $d$, $\log_B d \in \mathbb{Q}$, and $\langle \log_B d \rangle$ is not dense. What about $\langle \langle d^n \rangle \rangle$?
PT Thm: In that case certain $t$ are not achieved.
What about other bases?

Let $B \in \mathbb{N}$ be our base, and $D = \{1, 2, \ldots, B - 1\}$.

Kronecker: $x \in \mathbb{R} \setminus \mathbb{Q}$, if and only if $\langle x \rangle$ is dense in $\mathbb{R}/\mathbb{Z}$.
ERT technique: $\langle\langle x \rangle\rangle = \lfloor B^{\pi(\log_B x)} \rfloor$

If $B$ is not a perfect power, then $\log_B d \not\in \mathbb{Q}$, and the same argument works; i.e. all $t$ are achieved.

If $B$ is a perfect power, then for certain $d$, $\log_B d \in \mathbb{Q}$, and $\langle \log_B d \rangle$ is not dense. What about $\langle\langle d^n \rangle\rangle$?
PT Thm: In that case certain $t$ are not achieved.
What about other bases?

Let $B \in \mathbb{N}$ be our base, and $D = \{1, 2, \ldots, B - 1\}$.

Kronecker: $x \in \mathbb{R} \setminus \mathbb{Q}$, if and only if $\langle x \rangle$ is dense in $\mathbb{R}/\mathbb{Z}$.
ERT technique: $\langle\langle x \rangle\rangle = \lfloor B^\pi(\log_B x) \rfloor$

If $B$ is not a perfect power, then $\log_B d \notin \mathbb{Q}$, and the same argument works; i.e. all $t$ are achieved.
If $B$ is a perfect power, then for certain $d$, $\log_B d \in \mathbb{Q}$, and $\langle \log_B d \rangle$ is not dense. What about $\langle\langle d^n \rangle\rangle$?
PT Thm: In that case certain $t$ are not achieved.
What about other bases?

Let $B \in \mathbb{N}$ be our base, and $D = \{1, 2, \ldots, B - 1\}$.

Kronecker: $x \in \mathbb{R} \setminus \mathbb{Q}$, if and only if $\langle x \rangle$ is dense in $\mathbb{R}/\mathbb{Z}$.

ERT technique: $\langle \langle x \rangle \rangle = \lfloor B^\pi(\log_B x) \rfloor$

If $B$ is not a perfect power, then $\log_B d \not\in \mathbb{Q}$, and the same argument works; i.e. all $t$ are achieved.

If $B$ is a perfect power, then for certain $d$, $\log_B d \in \mathbb{Q}$, and $\langle \log_B d \rangle$ is not dense. What about $\langle \langle d^n \rangle \rangle$?

PT Thm: In that case certain $t$ are not achieved.
General Kronecker’s Theorem

Let \( x_1, x_2, \ldots, x_k \in \mathbb{R} \). Set \( y \) to be the natural projection of \((x_1, x_2, \ldots, x_k)\) into the additive group \( \mathbb{R}^n/\mathbb{Z}^n \). TFAE:

1. \( \{1, x_1, \ldots, x_k\} \) is \( \mathbb{Q} \)-linearly independent
2. \( \langle y \rangle \) is dense in \( \mathbb{R}^k/\mathbb{Z}^k \).

ERT: \( \log_{10} 2 + \log_{10} 5 = 1: \langle (\log_{10} 2, \log_{10} 5) \rangle \not\text{ dense} \) in \( \mathbb{R}^2/\mathbb{Z}^2 \). In particular, \( \langle \langle 2^n \rangle, \langle 5^n \rangle \rangle \neq (2, 5) \) for \( n \neq 1 \).
Results

- If $B$ is a perfect power, not all $t$ achieved for certain $d$.

- If $B$ is not a perfect power, all $t$ achieved for every digit $d$, singly.

- If $B = uv$ for $u, v > 1$ and $\gcd(u, v) = 1$, then $(\langle u^n \rangle, \langle v^n \rangle) \neq (u, v)$ for $n \neq 1$.
  Also, $(a, a, \ldots, a)$ is not achieved.

- If $B$ is a prime, work in progress.
Results

• If \( B \) is a perfect power, not all \( t \) achieved for certain \( d \).

• If \( B \) is not a perfect power, all \( t \) achieved for every digit \( d \), singly.

• If \( B = uv \) for \( u, v > 1 \) and \( \gcd(u, v) = 1 \), then \((⟨⟨u^n⟩⟩, ⟨⟨v^n⟩⟩) ≠ (u, v) \) for \( n ≠ 1 \).
  Also, \((a, a, \ldots, a)\) is not achieved.

• If \( B \) is a prime, work in progress.
Results

- If $B$ is a perfect power, not all $t$ achieved for certain $d$.

- If $B$ is not a perfect power, all $t$ achieved for every digit $d$, singly.

- If $B = uv$ for $u, v > 1$ and $\gcd(u, v) = 1$, then $(\langle u^n \rangle, \langle v^n \rangle) \neq (u, v)$ for $n \neq 1$.
  Also, $(a, a, \ldots, a)$ is not achieved.

- If $B$ is a prime, work in progress.
For Further Reading

Jaap Eising, David Radcliffe, Jaap Top
A Simple Answer to Gelfand’s Question
*American Mathematical Monthly* 122 (3) 2015, pp. 234-245.