A Generalization of Bonse’s Inequality

Over 100 years ago, Bonse proved (see [1]) his famous inequality, which relates the sequence of primes \( p_1 = 2, p_2 = 3, p_4 = 5, \ldots \) as:

For all \( n \geq 4 \), we have \( p_1 p_2 \cdots p_n > p_{n+1}^2 \).

We offer the following generalization.

**Theorem.** Suppose we have constants \( \mu, \lambda \) satisfying \( 1 < \mu \leq \lambda \). Let \( a_i \) be a nondecreasing sequence of real numbers satisfying \( a_1 = \mu \) and \( a_{i+1} \leq \lambda a_i \) (for each \( i \geq 1 \)). Then, taking \( K = 2 + 3 \log_{\mu} \lambda \), we get:

For all \( n \geq K \), we have \( a_1 a_2 \cdots a_n \geq a_{n+1}^2 \).

The inequality is strict except possibly for \( n = K \).

Note that Bertrand’s postulate gives \( p_{n+1} \leq 2p_n \), so we may take \( \mu = \lambda = 2 \) with \( a_n = p_n \), and recover Bonse’s inequality (apart from \( n = 4, 5 \)).

**Proof.** Note that \( K \) is chosen so that \( n \geq K \) implies \( \mu^{n-2} \geq \lambda^3 \). We get

\[
a_1 a_2 \cdots a_{n-1} \geq \mu^{n-2} a_{n-1} \geq \lambda^3 a_{n-1} \geq \lambda^2 a_n \geq \lambda a_{n+1} \geq \frac{a_{n+1}^2}{a_n}
\]

The first inequality follows from \( a_i \geq \mu \), and the last three each use \( a_{i+1} \leq \lambda a_i \).

Note that if \( n > K \) then \( \mu^{n-2} > \lambda^3 \), making the second inequality strict.

Note that equality is possible if \( n = K \in \mathbb{Z} \), via \( a_1 = a_2 = \cdots = a_{K-1} = \mu \), \( a_K = \lambda \mu \), and \( a_{K+1} = \lambda^2 \mu \). Now \( \mu^{n-2} = \lambda^3 \), so \( a_1 a_2 \cdots a_n = a_{n+1}^2 \).

**REFERENCES**


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