A Geometric mean–Arithmetic mean Ratio Limit

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One of the truly delightful results related to the natural numbers is the following limit of the ratio of the geometric and arithmetic means of the first \( n \) natural numbers:

\[
\lim_{n \to \infty} \sqrt[1/n]{} \frac{1 \cdot 2 \cdot 3 \cdots n}{\frac{1+2+3+\cdots+n}{n}} = \frac{2}{e}.
\]  

(1)

Obviously, the ratio in (1) approaches its limit really slowly. In fact, the relative difference between the ratio and its limiting value is of order, \((n+1)^{1/n}\), as \( n \to \infty \). For example, this is about 2 percent when \( n = 100 \).

Some generalisation of the limit can be found in [1]–[3]. In this note, we offer a short proof and generalisation of limit (1). Our result is narrower here, but the techniques are wholly different from [1], [2], and [3], and rely solely, in theory, on algebraic limit properties. Our proof relies on the following well-known result.

**Lemma.** [see, e.g., p.81 of [4]] Let \( a_n \) be a sequence of positive reals with \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = L \). Then \( \lim_{n \to \infty} \sqrt[n]{a_n} = L \).

We now establish a generalisation of (1) in the following theorem.

**Theorem.** Let \( \{b_n\} \) be a sequence of positive reals with \( \lim_{n \to \infty} b_n - n = 0 \). Then

\[
\lim_{n \to \infty} \sqrt[n]{b_1b_2b_3\cdots b_n} = \frac{2}{e}.
\]

Proof. We apply the lemma to \( a_n = (\prod_{i=1}^{n} b_i)^{1/n} / (\frac{1}{n} \sum_{i=1}^{n} b_i)^{1/n} \). Note that \( \frac{1}{n} \sum_{i=1}^{n} b_i = \frac{1}{n} \sum_{i=1}^{n} (b_i - i) + \frac{1}{n} \sum_{i=1}^{n} i \), and define \( c_n = \frac{1}{n} \sum_{i=1}^{n} (b_i - i) \).

\[
\frac{a_{n+1}}{a_n} = \frac{b_{n+1}}{b_n} \left( \frac{c_n + \frac{n+1}{2}}{c_{n+1} + \frac{n+2}{2}} \right)^n = \frac{b_{n+1}}{c_{n+1} + \frac{n+2}{2}} \left( \frac{n+1 + 2c_n}{n + 2 + 2c_{n+1}} \right)^n.
\]

Noting that \( \lim_{n \to \infty} c_n = 0 \), we see that the limit of the first part is 2. We may find the limit of the second part directly, or using the main result of [5]:

\[
\lim_{n \to \infty} \left( \frac{n+1 + 2c_n}{n + 2 + 2c_{n+1}} \right)^n = \exp \left( \lim_{n \to \infty} \frac{n+1 + 2c_n - 2c_{n+1}}{n + 2 + 2c_{n+1}} \right) = e^{-1}.
\]

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This completes the proof.

Note that taking $b_n = n$ in the theorem gives (1). As a general example, the theorem applies to any sequence $b_n = n + f(n)$, where $f(n) \to 0$. For example, $b_n = n + \frac{1}{\sqrt{n}}$.

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References


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