Wald’s Identity vs. Tail Sum Formula

Let $N$ be an arbitrary nonnegative integer-valued random variable with finite expectation $\mathbb{E}[N]$. Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed nonnegative random variables, independent of $N$, with $\mathbb{E}[X_1] < \infty$. Two well-known theorems in this context are Wald’s Identity,

$$\mathbb{E} \left[ \sum_{n=1}^{N} X_n \right] = \mathbb{E}[X_1]\mathbb{E}[N],$$

and the Tail Sum Formula

$$\mathbb{E}[N] = \sum_{n=1}^{\infty} \mathbb{P}(N \geq n).$$

We prove that these theorems are equivalent via the following calculation.

$$\mathbb{E} \left[ \sum_{n=1}^{N} X_n \right] = \mathbb{E} \left[ \sum_{n=1}^{\infty} X_n \mathbb{I}\{N \geq n\} \right] = \sum_{n=1}^{\infty} \mathbb{E}[X_n \mathbb{I}\{N \geq n\}] = \sum_{n=1}^{\infty} \mathbb{E}[X_n] \mathbb{P}(N \geq n).$$

Here the first equality is justified because $\sum_{n=1}^{N} X_n = \sum_{n=1}^{\infty} X_n \mathbb{I}\{N \geq n\}$; the second because the $X_i$’s are nonnegative; the third because $N$ is independent of the $X_i$’s; and the last because the $X_i$’s are identically distributed.

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