Summation Using an Expected Value Identity

Let \( X \) be a random variable taking values among the nonnegative integers. We recall the well-known identity \( \mathbb{E}[X] = \sum_{n \geq 0} p(X > n) \). For convenience, we set \( p_t = \mathbb{P}(X = t) \), arriving at the beautiful identity

\[
\sum_{x=0}^{\infty} xp_x = \sum_{x=0}^{\infty} \left( 1 - \sum_{y=0}^{x} p_y \right).
\]

If we further insist that \( X \) has support in \([0, n]\), we may rearrange this to get

\[
n + 1 = \sum_{x=0}^{n} 1 = \sum_{x=0}^{n} \left( x p_x + \sum_{y=0}^{x} p_y \right).
\]

We now offer two applications of (2). First, take the uniform distribution for \( X \), namely \( p_x = \frac{1}{n+1} \). This gives

\[
n + 1 = \sum_{x=0}^{n} \left( \frac{x}{n+1} + \sum_{y=0}^{x} \frac{1}{n+1} \right) = \sum_{x=0}^{n} \frac{x}{n+1} + \frac{x+1}{n+1} = 1 + \frac{2}{n+1} \sum_{x=0}^{n} x.
\]

We may rearrange this to find a new proof of the familiar \( \sum_{x=0}^{n} x = \frac{n(n+1)}{2} \). Compare this also to the probabilistic proof in [1]. For our second application, take the distribution \( p_x = \frac{2x}{n(n+1)} \). Applying (2) gives

\[
n + 1 = \sum_{x=0}^{n} \left( \frac{2x^2}{n(n+1)} + \sum_{y=0}^{x} \frac{2y}{n(n+1)} \right) = \frac{1}{n(n+1)} \sum_{x=0}^{n} 2x^2 + x(x+1).
\]

We rearrange this to \( 3 \sum_{x=0}^{n} x^2 = n(n+1)^2 - \sum_{x=0}^{n} x \), which gives a new proof of the familiar \( \sum_{x=0}^{n} x^2 = \frac{n(n+1)(2n+1)}{6} \).

REFERENCES

1. Treviño, E. (2019). Probabilistic Proof that \( 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \). Amer. Math. Monthly. 126 (9): 840.

—Submitted by Reza Farhadian, Razi University and Vadim Ponomarenko, San Diego State University

do\!\!i.org/10.XXXX/amer.math.monthly.122.XX.XXX

MSC: Primary 40C99, 60C99

January 2014]