A Statistical Proof of Chebyshev’s Sum Inequality

Let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be arbitrary real numbers satisfying $x_1 \geq x_2 \geq \cdots \geq x_n$ and $y_1 \geq y_2 \geq \cdots \geq y_n$. The well-known Chebyshev’s Sum Inequality states that

$$\frac{1}{n} \sum_{i=1}^{n} x_i y_i \geq \left( \frac{1}{n} \sum_{i=1}^{n} x_i \right) \left( \frac{1}{n} \sum_{i=1}^{n} y_i \right).$$

Typical proofs rely on clever algebra built upon the observation that the quantity $(x_i - x_j)(y_i - y_j)$ is nonnegative for all $i, j$. We offer a statistical proof. Let $\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$ and $\bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i$. Since the $x_i$’s and $y_i$’s decrease together, they have nonnegative covariance, i.e.,

$$0 \leq \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y}) = \frac{1}{n} \left( \sum_{i=1}^{n} x_i y_i - \bar{y} \sum_{i=1}^{n} x_i - \bar{x} \sum_{i=1}^{n} y_i + n \bar{x} \bar{y} \right)$$

$$= \frac{1}{n} \left( \sum_{i=1}^{n} x_i y_i - n \bar{x} \bar{y} - n \bar{x} \bar{y} + n \bar{x} \bar{y} \right) = \frac{1}{n} \left( \sum_{i=1}^{n} x_i y_i \right) - \bar{x} \bar{y}$$

Rearranging, we get $\frac{1}{n} \left( \sum_{i=1}^{n} x_i y_i \right) \geq \bar{x} \bar{y}$, as desired. If instead the real numbers satisfy $x_1 \leq x_2 \leq \cdots \leq x_n$ and $y_1 \geq y_2 \geq \cdots \geq y_n$, the covariance is nonpositive and the inequality is reversed.

—Submitted by Reza Farhadian, Razi University and Vadim Ponomarenko, San Diego State University