A Cauchy–Schwarz Type Inequality for Differences

Theorem. *Let* $u_1, u_2, \ldots, u_n$ *and* $w_1, w_2, \ldots, w_n$ *be real numbers satisfying* $u_i \geq w_i \geq 0$ *for* $i = 1, 2, \ldots, n$. *Then*

$$\sum_{i=1}^{n} \sqrt{u_i^2 - w_i^2} \leq \sqrt{\left(\sum_{i=1}^{n} u_i\right)^2 - \left(\sum_{i=1}^{n} w_i\right)^2}.$$

Proof. Let a, b be real numbers with $a \geq b \geq 0$. We can verify the algebraic identity $(a^2 - b^2)(u_i^2 - w_i^2) + (bu_i - aw_i)^2 = (au_i - bw_i)^2$. Since $(bu_i - aw_i)^2 \geq 0$, in fact $(a^2 - b^2)(u_i^2 - w_i^2) \leq (au_i - bw_i)^2$. Since $a \geq b \geq 0$ and $u_i \geq w_i \geq 0$, we may take square roots, getting

$$\sqrt{a^2 - b^2}\sqrt{u_i^2 - w_i^2} \leq au_i - bw_i.$$

We now sum over $i = 1, 2, \ldots n$ and set $a = \sum_{i=1}^{n} u_i$, $b = \sum_{i=1}^{n} w_i$. This gives

$$\sqrt{a^2 - b^2}\sum_{i=1}^{n} \sqrt{u_i^2 - w_i^2} \leq a^2 - b^2.$$ ■

—Submitted by Reza Farhadian, Lorestan University and
Vadim Ponomarenko, San Diego State University

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