A Cauchy–Schwarz Type Inequality for Differences

**Theorem.** Let \( u_1, u_2, \ldots, u_n \) and \( w_1, w_2, \ldots, w_n \) be real numbers satisfying \( u_i \geq w_i \geq 0 \) for \( i = 1, 2, \ldots, n \). Then

\[
\sum_{i=1}^{n} \sqrt{u_i^2 - w_i^2} \leq \sqrt{\left( \sum_{i=1}^{n} u_i \right)^2 - \left( \sum_{i=1}^{n} w_i \right)^2}.
\]

**Proof.** Let \( a, b \) be real numbers with \( a \geq b \geq 0 \). We can verify the algebraic identity \((a^2 - b^2)(u_i^2 - w_i^2) + (bu_i - aw_i)^2 = (au_i - bw_i)^2\). Since \((bu_i - aw_i)^2 \geq 0\), in fact \((a^2 - b^2)(u_i^2 - w_i^2) \leq (au_i - bw_i)^2\). Since \( a \geq b \geq 0 \) and \( u_i \geq w_i \geq 0 \), we may take square roots, getting

\[
\sqrt{a^2 - b^2} \sqrt{u_i^2 - w_i^2} \leq au_i - bw_i.
\]

We now sum over \( i = 1, 2, \ldots, n \) and set \( a = \sum_{i=1}^{n} u_i, b = \sum_{i=1}^{n} w_i \). This gives

\[
\sqrt{a^2 - b^2} \sum_{i=1}^{n} \sqrt{u_i^2 - w_i^2} \leq a^2 - b^2.
\]

—Submitted by Reza Farhadian, Lorestan University and Vadim Ponomarenko, San Diego State University

doi.org/10.XXXX/amer.math.monthly.122.XX.XXX
MSC: Primary 26D15