DIVERSITY IN INSIDE FACTORIAL MONOIDS

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Abstract. We apply the recently introduced monoid invariant "diversity" to inside factorial monoids. In this context, the diversity of an element counts the number of its different almost primary components. Inside factorial monoids are characterized via diversity and strong homogeneity. A new invariant complementary to diversity, height, is introduced. These two invariants are connected with the well-known invariant of elasticity.

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1. Introduction

The Fundamental Theorem of Arithmetic tells us that every natural number different from 1 can be written in a unique manner (up to reindexing) as the product of different primes each taken to some power. This is, of course, no longer possible for arbitrary monoids, but under certain assumptions, one can retain some features of this unique representation. Applying the concept of diversity, developed in [8] to inside factorial monoids, as introduced in [7], we mimic the "number of different prime factors" of an element by its diversity. As for the "powers of primes", we introduce the concept of height. Both diversity and height are useful complementary concepts to analyze inside factorial monoids, which include many interesting non-factorial monoids: for example, all principal orders of algebraic number fields and, more generally, Krull monoids with torsion class group. The factorial monoids then turn out to be the limit case where atomic diversity (Definition 1.3) as well as the height of the monoid (Definition 4.6) are both equal to 1.

In this paper, all monoids under consideration are commutative and cancellative. Unless otherwise stated, our monoids will be written multiplicatively with identity denoted by 1. If M is a monoid, then $M^{\times}$ denotes the set of units (or invertible elements) of M, and we use $M^\bullet$ to denote $M \setminus M^\times$. If $\pi \in M^\bullet$, we say that π is an **atom** (or **irreducible**) element of M if, for all $a, b \in M$ with $\pi = ab$, we have $a \in M^\times$ or $b \in M^\times$. The set of atoms of M will be denoted by $\mathcal{A}(M)$, and we say that M is **atomic** if every nonunit element of M can be written as a product of atoms. If S is a nonempty finite subset of M, then by $\prod S$, we mean the product of the elements of S. If S is empty, then by $\prod S$ we mean 1. We use $\mathbb{N}$ to denote the set of positive integers, and $\mathbb{N}_0$ to denote the set of nonnegative integers.

If A and B are nonempty subsets of a monoid M, then by AB, we mean $\{ab : a \in A, b \in B\}$, and we denote $\{x\}A$ by xA. A subset I of M is called an **ideal** of M if $IM = I$, and if I is an ideal, we say I is a **prime ideal** of M if whenever $ab \in I$ for $a, b \in M$, then $a \in I$ or $b \in I$.

If I is an ideal of M, we define the **radical** of I, denoted by $\sqrt{I}$, to be

$$\sqrt{I} = \{x \in M : x^n \in I \text{ for some } n \in \mathbb{N}\}.$$

As is the case with rings, we have $\sqrt{IJ} = \sqrt{I} \cap \sqrt{J}$ for any ideals I and J of M. It is apparent that $q \in M$ is almost primary if and only if $\sqrt{qM}$ is a prime ideal of M. For proofs of the preceding assertions regarding monoid ideals, the reader is referred to [4].[1]

Recently, much attention has been paid to factorization theory in (commutative, cancellative) monoids, and in particular to factorization in integral domains. Although most monoids do not have the property of unique factorization, examples abound of monoids where each element has some power with a unique representation: these are the inside factorial monoids introduced by the first author in [7].

Definition 1.1. Let M be a monoid. We say that $Q \subseteq M^\bullet$ is a **Cale basis** for M if for each $x \in M$, the following conditions hold:

(i) There exists $u(x) \in M^\times$, $n(x) \in \mathbb{N}$, and $\{t(x,q)\}_{q \in Q} \in \mathbb{N}_0^Q$ with only finitely many of the $t(x,q)$ nonzero, and

$$x^{n(x)} = u(x) \prod_{q \in Q} q^{t(x,q)}.$$

(ii) For all $k \in \mathbb{N}$, if $x^k = u \prod_{q \in Q} q^{t_q} = v \prod_{q \in Q} q^{s_q}$ for some $u, v \in M^\times$ and $t_q, s_q \in \mathbb{N}_0^Q$ with only finitely many of the t_q and s_q nonzero, then $u = v$ and $t_q = s_q$ for all $q \in Q$.

If there exists a Cale basis of M, we say that M is **inside factorial**.

[1]Note that our concept of an ideal is really that of an s-ideal as defined in [4].

If Q is a Cale basis of M, for each $x \in M^{\bullet}$, we denote by $m(x)$ the smallest value of $n(x)$ satisfying (i) above, and we denote by $x(q)$ the uniquely determined $t(x,q)$ corresponding to $m(x)$. Further, we define the **support** of x, denoted $Supp(x)$, to be $Supp(x) = \{q \in Q : x(q) > 0\}$.

As an example, consider the Hilbert monoid $H = 1 + 4\mathbb{N}_0 = \{n \in \mathbb{N} : n \equiv 1 \mod (4)\}$ (a multiplicative submonoid of $\mathbb{N}$). We have the following non-unique factorization of 441 into irreducibles:

$$441 = 21 \cdot 21 = 9 \cdot 49.$$

However, squaring 441, we see that

$$21^2 21^2 = 9^2 49^2,$$

and we can rewrite both sides of the above equation as $3^2 3^2 7^2 7^2$. This argument can be generalized to show that H is an inside factorial monoid, and that $\{p^2 : p \in \mathbb{N} \text{ is prime and } p \equiv 3 \mod (4)\}$ is a Cale basis for H. More generally, any Krull monoid with torsion class group is an inside factorial monoid (cf. [3]).

Closely related to the concept of inside factorial monoids is that of the extraction degree, as introduced in [5].

Definition 1.2. Let M be a monoid. The function $\lambda : M \times M \to [0, \infty]$ defined by

$$\lambda(x,y) = \sup\left\{\frac{m}{n} : m \in \mathbb{N}_0, n \in \mathbb{N}, \text{ and } x^m | y^n\right\}$$

is called the **extraction degree** on M. If for all $x, y \in M^{\bullet}$ there exist $m \in \mathbb{N}_0$ and $n \in \mathbb{N}$ such that $x^m | y^n$ and $\lambda(x,y) = \frac{m}{n}$, then we call M an **extraction monoid**.

It has been proven that any inside factorial monoid is an extraction monoid (cf. [2]).

If M is a monoid and $q \in M$ is a nonunit, we say that q is **almost primary** if whenever $q|ab$ for $a, b \in M$, then there exists $k \in \mathbb{N}$ such that $q|a^k$ or $q|b^k$. It has been shown that if M is inside factorial with Cale basis Q, then every element of Q is almost primary. In [8], the second and third authors introduced a monoid invariant (diversity) that generalizes this property of almost primary, as well as two conditions (homogeneity and strong homogeneity) that lie between almost primary and primary.

Definition 1.3. Let M be a monoid.

(1) We say that $x|S$ (in M) if $x \in M$, S is a finite subset of M, and if there exists $t \in \mathbb{N}$ such that $x| \left(\prod S\right)^t$.

(2) We say that x **strictly divides** S, denoted $x\|S$, if $x|S$ and $x \nmid T$ for all $T \subsetneq S$.
(3) We define the **diversity** of x, denoted $\mathrm{div}(x)$, to be

$$\mathrm{div}(x) = \sup\{|S| : S \subseteq M \text{ with } x\|S\}.$$

(4) We define the **diversity** of M and the **atomic diversity** of M, denoted by $\mathrm{div}(M)$ and $\mathrm{div}_{\mathrm{a}}(M)$, respectively, by

$$\mathrm{div}(M) = \sup_{x\in M} \mathrm{div}(x), \text{ and } \mathrm{div}_{\mathrm{a}}(M) = \sup_{x\in\mathcal{A}(M)} \mathrm{div}(x).$$

(5) We say that $x \in M^\bullet$ is **homogeneous** if $\mathrm{div}(x) = 1$ and for all $y \in M^\bullet$ with $y|\{x\}$, we have $x|\{y\}$.
(6) We say that $x \in M^\bullet$ is **strongly homogeneous** if $\mathrm{div}(x) = 1$ and for all $y \in M^\bullet$ and $S \subseteq M$, with $y\|S$ and $x \in S$, we have $x|\{y\}$.

Every strongly homogeneous element of M is clearly homogeneous, but not conversely (cf. [8, Example 3.7]).

For the sake of completeness, we recall some results and a definition from [8] that we will put to use.

Proposition 1.4. *Let M be a monoid, and let $x, y \in M$. Then:*

(1) *$\mathrm{div}(xy) \leq \mathrm{div}(x) + \mathrm{div}(y)$.*
(2) *$\mathrm{div}(x) = 0$ if and only if $x \in M^\times$.*
(3) *$\mathrm{div}(x) = 1$ if and only if x is almost primary.*
(4) *For all $n \in \mathbb{N}$, $\mathrm{div}(x) = \mathrm{div}(x^n)$.*

Proposition 1.5. *Let M be a monoid, and let $x \in M^\bullet$. Then x is homogeneous if and only if $\sqrt{xM}$ is a prime ideal that is maximal amongst radicals of proper principal ideals.*

Theorem 1.6. *Let M be a monoid and let $x \in M^\bullet$. If there exists a set of strongly homogeneous elements S such that $x\|S$, then $\mathrm{div}(x) = |S|$.*

Definition 1.7. Let M be a monoid, let $x \in M$, let $q_1, q_2, \cdots, q_t \in M$ be almost primary, and suppose that

$$x = q_1 q_2 \cdots q_t.$$

We say that the above factorization is a **reduced** factorization of x (into almost primary elements) if, for all $i \neq j$, $\sqrt{q_i M}$ and $\sqrt{q_j M}$ are incomparable.

In Section 2, we use the above results to study inside factorial monoids. Along the way, we prove a useful lemma (dubbed the "Cale exchange lemma") that characterizes when we can trade an element q_0 in a Cale basis Q for an element $a \in M^\bullet \setminus Q$ and still obtain a Cale basis (Lemma 2.2). We also prove that for an inside factorial monoid M, every almost primary element of M is strongly homogeneous (Theorem 2.3) and that for all $x \in M^\bullet$, $\mathrm{div}(x) = |Supp(x)|$ (Theorem 2.4).

In Section 3, we use diversity , strong homogeneity, and the results of Section 2 to give three characterizations of inside factorial monoids (Theorem 3.2).

In Section 4, we focus on atomic inside factorial monoids. We introduce new invariants, width and height, of atomic inside factorial monoids, and use these to find bounds on the elasticity.

In Section 5, we close with three examples illustrating the differences between width, height, and diversity.

2. Preliminary results

If x is a nonunit element of a monoid M, then any factorization of x into almost primary elements can be made into a reduced factorization, as shown by the following proposition.

Proposition 2.1. *Let M be a monoid, let $x \in M^\bullet$, and let $q_1, q_2, \cdots, q_t \in M$ be almost primary. Then:*

1. *If $x = q_1 q_2 \cdots q_t$, then there exists a reduced factorization $q'_1 q'_2 \cdots q'_s$ of x into almost primary elements such that for all i, there exists a j with $q_i | q'_j$.*
2. *If $\mathrm{div}(x) = t$ and if there exists $n \in \mathbb{N}$ such that $x^n = q_1 q_2 \cdots q_t$, then $q_1 q_2 \cdots q_t$ is a reduced factorization of x^n.*

Proof. 1. If $q_1 q_2 \cdots q_t$ is a reduced factorization of x into almost primary elements, then there is nothing to prove. Otherwise, assume (without loss of generality) that $\sqrt{q_1 M} \subseteq \sqrt{q_2 M}$. Letting $q = q_1 q_2$, we see that q is almost primary (as $\sqrt{qM} = \sqrt{q_1 q_2 M} = \sqrt{q_1 M} \cap \sqrt{q_2 M} = \sqrt{q_1 M}$ is a prime ideal of M). Therefore $x = q q_3 \cdots q_t$, and the result follows by induction.

2. Again, suppose (without loss of generality) that $\sqrt{q_1 M} \subseteq \sqrt{q_2 M}$. Letting $q = q_1 q_2$, it follows that q is almost primary. This implies that $t = \mathrm{div}(x) = \mathrm{div}(q q_3 q_4 \cdots q_t) \leq \mathrm{div}(q) + \mathrm{div}(q_3) + \mathrm{div}(q_4) + \cdots + \mathrm{div}(q_t) = t - 1$, a contradiction. $\square$

We now begin to apply our results and concepts thus far to inside factorial monoids. If M is an inside factorial monoid with Cale basis Q, and if S is a finite subset of M, then by $Supp(S)$, we mean $\bigcup_{s \in S} Supp(s)$. Further, we recall that for a monoid M, $x \in M^\bullet$ is **almost irreducible** if given $y \in M^\bullet$ with $y|x$, there exist $m, n \in \mathbb{N}$ and $u \in M^\times$ such that $y^m = u x^n$.

Lemma 2.2 (Cale Exchange Lemma)**.** *Let M be an inside factorial monoid with Cale basis Q. Pick $a \in M^\bullet \setminus Q$ and $q_0 \in Q$. Then, $(Q \setminus \{q_0\}) \cup \{a\}$ is again a Cale basis of M if and only if $a^k = uq_0^l$ for some $k, l \in \mathbb{N}$ and $u \in M^\times$.*

Proof. We set $Q' = (Q \setminus \{q_0\}) \cup \{a\}$.

($\Rightarrow$) If $x \in M^\bullet$, we will denote the support of x with respect to Q by $Supp_Q(x)$ and the support of x with respect to Q' by $Supp_{Q'}(x)$; likewise, we denote by $m'(x)$ the smallest power of x that has a Cale representation with respect to Q'. Note that since $a \in Q'$, $m'(a) = 1$.

Assume that $q_0 \notin Supp_Q(a)$. Then,

$$a^{m(a)} = u(a) \prod_{q \in Q} q^{a(q)}, \text{ hence } a^{m'(a)m(a)} = u(a)^{m'(a)} \prod_{q \in Q' \setminus \{a\}} q^{a(q)m'(a)}$$

and a power of $a^{m'(a)}$ has two different Cale representations with respect to Q', a contradiction. Thus, $q_0 \in Supp_Q(a)$.

Now, since a is in a Cale basis, a is almost primary ([2, Lemma 2]). Looking at the Cale representation of a with respect to Q, we see that there exists $q \in Supp_Q(a)$ with $a | q^c$ for some $c \in \mathbb{N}$. If $q \neq q_0$, then $q_0^{a(q_0)}$ divides $q^{cm(a)}$, violating uniqueness of Cale representation with respect to Q. Therefore $Supp_Q(a) = \{q_0\}$. It follows that $a^{m(a)} = u(a)q_0^{a(q_0)}$.

($\Leftarrow$) Let $x \in M^\bullet$. If $q_0 \notin Supp(x)$, then a power of x is an associate of a product of elements from $(Q \setminus \{q_0\}) \cup \{a\}$. Otherwise, we have

$$x^{m(x)} = u(x) q_0^{t(x,q_0)} \prod_{q \in Q \setminus \{q_0\}} q^{t(x,q)}$$

(where $t(x, q_0) > 0$). Raising both sides to the l power, we have

$$x^{lm(x)} = u(x)^l u^{-t(x,q_0)} a^{kt(x,q_0)} \prod_{q \in Q \setminus \{q_0\}} q^{lt(x,q)},$$

satisfying Definition 1.1(i). Next, suppose

$$x^n = u_i(x) a^{t_i(x,a)} \prod_{q \in Q \setminus \{q_0\}} q^{t_i(x,q)}$$

for $i = 1, 2$. Raising both sides to the k power, we have

$$x^{kn} = u_i(x)^k u^{t_i(x,a)} q_0^{lt_i(x,a)} \prod_{q \in Q \setminus \{q_0\}} q^{kt_i(x,q)}.$$

By the uniqueness of Cale representation in Q, we obtain Definition 1.1(ii). Therefore Q' is a Cale basis of M. □

Theorem 2.3. *Let M be an inside factorial monoid with Cale basis Q, let $x \in M^\bullet$, and let S be a finite subset of M. Then:*

(1) *If x is almost primary, then $x^{m(x)} = u(x)q_0^{x(q_0)}$ for some $q_0 \in Q$.*
(2) *If x is almost primary, then every power of x is almost irreducible.*
(3) *$Supp(S) = Supp\left(\prod S\right)$.*
(4) *$x|S$ if and only if $Supp(x) \subseteq Supp(S)$.*
(5) *Every almost primary element of M is strongly homogeneous.*

Proof. 1. Let $x \in M$ be almost primary, and let $x^{m(x)} = u(x)\prod_{q\in Q} q^{x(q)}$ be the Cale representation of x. Since x is almost primary, there exists $q_0 \in Supp(x)$ and $k \in \mathbb{N}$ such that $x|q_0^k$. Writing $xr = q_0^k$, and $r^{m(r)} = u(r)\prod_{q\in Q} q^{r(q)}$, we see that

$$(xr)^{m(x)\cdot m(r)} = u(x)^{m(r)}u(r)^{m(x)} \prod_{q\in Q} q^{x(q)m(r)+r(q)m(x)} = q_0^{k\cdot m(x)\cdot m(r)}.$$

By uniqueness of Cale representation, we see that $Supp(x) = Supp(r) = \{q_0\}$.

2. If x is almost primary, then, by 1, $Supp(x) = \{q_0\}$ for some $q_0 \in Q$. It follows that for all $y \in M^\bullet$ and $k \in \mathbb{N}$, $y|x^k$ implies $Supp(y) = \{q_0\}$. Therefore $y^m = ux^{kn}$ for some $m, n \in \mathbb{N}$ and $u \in M^\times$.

3. Let $r = m\left(\prod S\right)\left(\prod_{s\in S} m(s)\right)$. We note that $\left(\prod S\right)^r$ is a power of $\left(\prod S\right)^{m(\prod S)}$, but is also a product of powers of $s^{m(s)}$, for each $s \in S$. By uniqueness of Cale representation (using the same argument as in 1), the supports of $\prod S$ and S must be the same.

4. Set $z = \prod S$. Assume that $x|S$. Then $xr = z^k$ for some $r \in M$ and $k \in \mathbb{N}$, and hence, by 3, $Supp(x) \subseteq Supp(\{x, r\}) = Supp(xr) = Supp(z^k) = Supp(z) = Supp(S)$. On the other hand, assume that $Supp(x) \subseteq Supp(S) = Supp(z)$. Set $t = m(z) \cdot \max\{x(q)\,|q \in Supp(x)\}$. Then, there exists $w \in M^\times$ such that

$$\frac{z^t}{x^{m(x)}} = w \prod_{q\in Supp(x)} q^{z(q)\cdot(t/m(z))-x(q)} \in M.$$

Hence $x^{m(x)}|z^t$ and thus $x|S$.

5. Pick almost primary $y \in M^\bullet$ and $S \subseteq M$ such that $y\|S$ and $S = \{x, s_1, s_2, \cdots, s_k\}$. By 1, there is some $q_0 \in Q$ with $Supp(x) = \{q_0\}$, and using 4, $Supp(y) \subseteq Supp(S)$, and in particular, $y|\{q_0, s_1, s_2, \cdots, s_k\}$. If $q_0 \in Supp(y)$, then $x|\{y\}$. Otherwise, by 4, $Supp(y) \subseteq \{s_1, s_2, \cdots, s_k\}$, contradicting the fact that $y\|S$. $\square$

Theorem 2.4. *Let M be an inside factorial monoid with a Cale basis Q. For $x \in M^\bullet$, $div(x) = |Supp(x)|$.*

Proof. By uniqueness of Cale representation, we have that $x \| Supp(x)$, implying that $\operatorname{div}(x) \geq |Supp(x)|$. On the other hand, writing $Supp(x) = \{q_1, q_2, \cdots, q_m\}$ and $x^{m(x)} = u(x) q_1^{x(q_1)} q_2^{x(q_2)} \cdots q_m^{x(q_m)}$, we find that $\operatorname{div}(x) = \operatorname{div}(x^{m(x)}) = \operatorname{div}(q_1^{x(q_1)} \cdots q_m^{x(q_m)}) \leq \operatorname{div}(q_1^{x(q_1)}) + \operatorname{div}(q_2^{x(q_2)}) + \cdots + \operatorname{div}(q_m^{x(q_m)}) = |Supp(x)|$. $\square$

We remark that Theorem 2.4 leads to an alternate proof of [2, Cor. 2], characterizing all Cale bases in an inside factorial monoid.

Corollary 2.5. *Let M be an inside factorial monoid with Cale basis Q. Then every element of M has finite diversity and div$(M) = |Q|$. Furthermore, if $S \subseteq M$ is a finite set of atoms such that no element of S divides any power of the product of the subsequent elements of S, then div $(\prod S) = \sum_{s \in S} \operatorname{div}(s)$.*

3. A characterization of inside factorial monoids

Following [7], a monoid M is said to be of **finite type** if M satisfies the ascending chain condition on radicals of principal ideals; i.e. given $x_1, x_2, \cdots \in M$ with

$$\sqrt{x_1 M} \subseteq \sqrt{x_2 M} \subseteq \cdots \subseteq \sqrt{x_n M} \subseteq \cdots,$$

there exists $N \in \mathbb{N}$ such that for all $m \geq N$, $\sqrt{x_N M} = \sqrt{x_m M}$. Also, given nonunits $x, y \in M$, we say y is a **component** of x if $y | x^n$ for some $n \in \mathbb{N}$ (or, equivalently, if $\sqrt{xM} \subseteq \sqrt{yM}$). With this terminology, we record the following theorem, which we will put to use momentarily.

Theorem 3.1 (Theorem 1 of [7])**.** *Let M be an extraction monoid of finite type, and let $A \subseteq M$ such that every nonunit in M has some component in A. Then, given any $x \in M$, some power of x is contained in a factorial monoid generated by a finite subset of A.*

Theorem 3.2. *Let M be a monoid. Then the following are equivalent:*

(i) *M is an inside factorial monoid.*
(ii) *For every $x \in M^\bullet$ with div$(x) \geq 2$, there exist $n \in \mathbb{N}$ and $y, z \in M^\bullet$ such that $x^n = yz$ and div$(x) =$ div$(x^n) =$ div$(y) +$ div(z). What is more, given an almost primary element $q \in M$, q is strongly homogeneous and every power of q is almost irreducible.*
(iii) *For every $x \in M^\bullet$, there exists $n \in \mathbb{N}$ such that $x^n = q_1 q_2 \cdots q_t$, where each q_i is strongly homogeneous and every power of each q_i is almost irreducible.*
(iv) *M is an extraction monoid, and given any $x \in M^\bullet$, there exists a set of strongly homogeneous elements S such that $x \| S$.*
(v) *M is an extraction monoid of finite type, and every nonunit of M has an almost primary component.*

Proof. ($(i) \Rightarrow (ii)$) Pick $x \in M$ with $\operatorname{div}(x) \geq 2$. Let M have Cale basis Q. Then, we have $x^{m(x)} = u(x) \prod_{q \in Q} q^{x(q)}$, and by Theorem 2.4, $x \| Supp(x)$ and $|Supp(x)| = \operatorname{div}(x)$. Therefore $\operatorname{div}(x) = \sum_{q \in Supp(x)} \operatorname{div}(q)$. The rest of this implication follows by Theorem 2.3.

($(ii) \Rightarrow (iii)$) Pick $x \in M^{\bullet}$. If $\operatorname{div}(x) = 1$, there is nothing to prove, so assume that $\operatorname{div}(x) \geq 2$. By hypothesis, there exist nonunits $y, z \in M$ and $n \in \mathbb{N}$ such that $x^n = yz$, with $\operatorname{div}(x) = \operatorname{div}(y) + \operatorname{div}(z)$. By induction, there exist $m, k \in \mathbb{N}$ and almost primary elements $y_1, y_2, \cdots, y_r, z_1, z_2, \cdots, z_s$ such that $y^m = y_1 y_2 \cdots y_r$, $z^k = z_1 z_2 \cdots z_s$, $\operatorname{div}(y) = r$, and $\operatorname{div}(z) = s$. Then, $x^{nmk} = y^{mk} z^{mk} = y_1^k y_2^k \cdots y_r^k z_1^m z_2^m \cdots z_s^m$. We note that each y_i^k and z_j^m is almost primary. Thus, we have written a power of x as a product of almost primary elements. All other assertions carry over directly from (ii).

($(iii) \Rightarrow (iv)$) We first prove that, under this assumption, if x is almost primary then x is strongly homogeneous. Suppose $x^n = q_1 q_2 \cdots q_t$ for some $n \in \mathbb{N}$ and strongly homogeneous q_i. Since x is almost primary, $x | \{q_i\}$ for some i. Since q_i is almost irreducible, there exist $m, k \in \mathbb{N}$ and $u \in M^{\times}$ such that $x^m = u q_i^k$. By Theorem 3.8 (5) from [8], q_i^k is strongly homogeneous, hence x is.

Let $x, y \in M^{\bullet}$. Suppose first that there are no $m, n \in \mathbb{N}$ such that $x^m | y^n$. Then $\lambda(x, y) = 0/1 = 0$ and $x^0 | y^1$.

Otherwise, pick any $m, n \in \mathbb{N}$ such that $x^m | y^n$. Applying (iii) twice, we have $x^u = q_1 q_2 \cdots q_t$ and $y^v = r_1 r_2 \cdots r_s$ for $u, v \in \mathbb{N}$ and q_i, r_j strongly homogeneous. Without loss of generality, we may also assume that the factorizations of x^u and y^v above are reduced. The q_i remain strongly homogeneous by the above. Since $x^m | y^n$, we have $x^{uvm} | y^{uvn}$, and hence $(q_1 q_2 \cdots q_t)^{vm} | (r_1 r_2 \cdots r_s)^{un}$. Since all the q_i and r_j are strongly homogeneous, we have $t \leq s$ and (without loss of generality) $\sqrt{q_i M} = \sqrt{r_i M}$ for each $1 \leq i \leq t$. By (iii), there exist, for each $1 \leq i \leq t$, $u_i \in M^{\times}$ and $(a_i, b_i) \in \mathbb{N} \times \mathbb{N}$ such that $q_i^{a_i} = u_i r_i^{b_i}$.

It follows that $\lambda(q_i, r_i) = \frac{a_i}{b_i}$. Without loss of generality, we may assume that $\frac{a_1}{b_1} \leq \frac{a_2}{b_2} \leq \cdots \leq \frac{a_t}{b_t}$. Set $N = \prod_{i=1}^{n} a_i b_i$. We now show that $x^{ua_1 N} | y^{vb_1 N}$. We have the following (for some $w \in M^{\times}$):

$$y^{vb_1 N} = \prod_{i=1}^{s} r_i^{b_1 N} = \left(\prod_{i=1}^{t} (r_i^{b_i})^{b_1 \frac{N}{b_i}} \right) \left(\prod_{i=t+1}^{s} r_i^{b_1 N} \right)$$

$$= w x^{ua_1 N} \left(\prod_{i=1}^{t} q_i^{(b_1 N a_i / b_i) - a_1 N} \right) \left(\prod_{i=t+1}^{s} r_i^{b_1 N} \right).$$

Note that $(b_1 N a_i / b_i) - a_1 N \in \mathbb{N}_0$, since $\frac{a_i}{b_i} \geq \frac{a_1}{b_1}$. Hence, $\lambda(x, y) \geq \frac{ua_1 N}{vb_1 N} = \frac{ua_1}{vb_1}$. In fact, we have equality; suppose that $x^m | y^n$ for some $m, n \in \mathbb{N}$. Again, we have

$x^{uvm}|y^{uvn}$, hence there exists $c \in M$ such that $(q_1q_2\cdots q_t)^{vm}c = (r_1r_2\cdots r_s)^{un}$. We have $q_i^{a_i} = u_ir_i^{b_i}$ for $u_i \in M^\times$ and $a_i, b_i \in \mathbb{N}$. Letting $A = \prod_{i=1}^t a_i$, and $A_i = A/a_i$ (for $1 \le i \le t$), we have

$$\prod_{i=1}^{s} r_i^{unA} = c^A \prod_{i=1}^{t} q_i^{vmA} = c^A \prod_{i=1}^{t} q_i^{a_iA_ivm} = c^A \prod_{i=1}^{t} u_i^{A_ivm} r_i^{b_iA_ivm}$$

Note that we cannot have $unA < b_1A_1vm$, otherwise $r_1|\prod_{i=2}^s r_i^{unA}$, but for each $2 \le i \le s$, r_1 divides no power of r_i, contradicting the fact that r_1 is almost primary. We conclude that $b_1A_1vm \le unA$, implying $b_1vm \le una_1$, and $\frac{m}{n} \le \frac{ua_1}{vb_1}$. Therefore $\lambda(x,y) = \frac{ua_1}{vb_1}$, and M is an extraction monoid.

Finally, for $x \in M^\bullet$, apply (iii) to get $x\|\{q_1, q_2, \cdots, q_t\}$, a set of strongly homogeneous elements.

($(iv) \Rightarrow (v)$) We will first show that M is of finite type. Suppose that we have $x_1, x_2, \cdots \in M$ with $\sqrt{x_1M} \subseteq \sqrt{x_2M} \subseteq \sqrt{x_3M} \subseteq \cdots$. Concentrating for a moment on x_1 and x_2, pick $r \in M$ and $a \in \mathbb{N}$ such that $x_2r = x_1^a$. If x_1 or x_2 is a unit, there is nothing to prove. So, picking a set $S = \{s_1, s_2, \cdots, s_t\}$ of strongly homogeneous elements of M with $x_1\|S$, we see that $x_2|S$. By Theorem 1.6, $\mathrm{div}(x_2) \le t$; we also have $t = \mathrm{div}(x_1)$. If $\mathrm{div}(x_2) = t$, then $x_2\|\{s_1, s_2, \cdots, s_t\}$ and since each s_i is strongly homogeneous, $s_i|\{x_2\}$ implying that $s_1s_2\cdots s_t|\{x_2\}$. However, the fact that $x_1|\{s_1, s_2, \cdots, s_t\}$ implies that $x_1|\{x_2\}$, and $\sqrt{x_1M} = \sqrt{x_2M}$.

Thus, generalizing the above argument from 1 and 2 to i and $i+1$, the only way for strict containment to hold between $\sqrt{x_iM}$ and $\sqrt{x_{i+1}M}$ is for $\mathrm{div}(x_{i+1}) < \mathrm{div}(x_i)$. Thus, the chain must stabilize, and M is of finite type.

Further, given $x \in M^\times$, we have $x\|S$ for some set S of strongly homogeneous elements. Thus, for any $s \in S$, $s|\{x\}$, and s is a component of x.

($(v) \Rightarrow (i)$) Let A be the set of almost primary elements of M, and choose (using Zorn's Lemma) a subset Q of A maximal with respect to the following property: If $q_1, q_2 \in Q$ are distinct, then $\sqrt{q_1M}$ and $\sqrt{q_2M}$ are incomparable. As M is an extraction monoid of finite type, and since every nonunit has a component in Q, Theorem 3.1 applies. Therefore, given any $x \in M^\bullet$, there exists $u \in M^\times$, $q_1, q_2, \cdots, q_n \in Q$ and $t_1, t_2, \cdots, t_n \in \mathbb{N}$ such that $x^n = uq_1^{t_1}q_2^{t_2}\cdots q_n^{t_n}$. Suppose we also have $x^n = vp_1^{a_1}p_2^{a_2}\cdots p_l^{a_l}$, where $v \in M^\times$, $a_i \in \mathbb{N}$, and $p_i \in Q$. As $q_1^{t_1}$ is almost primary, we see that (without loss of generality) $q_1^{t_1}|p_1^{a_1b}$ for some $b \in \mathbb{N}$. But then $\sqrt{p_1M} \subseteq \sqrt{q_1M}$, and by construction of Q, $\sqrt{q_1M} = \sqrt{p_1M}$ and $q_1 = p_1$. If $t_1 - a_1 > 0$, then $uq_1^{t_1-a_1}q_2^{t_2}\cdots q_n^{t_n} = vp_2^{a_2}p_3^{a_3}\cdots p_l^{a_l}$, and, for some $2 \le i \le l$, $q_1|p_i^{a_ic}$ for some $c \in \mathbb{N}$. However, we then have $\sqrt{p_iM} \subseteq \sqrt{q_1M} = \sqrt{p_1M}$, a contradiction. Therefore $t_1 \le a_1$, and by a similar argument, $t_1 \ge a_1$. Thus, canceling $q_1^{t_1}$ and $p_1^{a_1}$, we apply induction and see that $n = l$ and (without loss of generality) $q_i^{t_i} = p_i^{a_i}$ and $u = v$. We conclude that Q is a Cale basis for M. $\square$

4. Atomic inside factorial monoids

We now focus specifically on atomic inside factorial domains, and begin by characterizing those inside factorial monoids that have a Cale basis consisting of atoms.

Lemma 4.1. *Let M be an inside factorial monoid. Then M possesses a Cale basis consisting of atoms if and only if for each element in $M^\bullet$, there exists an atom dividing a power of this element. In particular, if M is atomic, then M possesses a Cale basis consisting of atoms.*

Proof. Obviously, if Q is a Cale basis of atoms, then for each element in $M^\bullet$ there exists a power of that element that is divided by an atom. Suppose, conversely, that the latter property holds. Let Q be a Cale basis of M and pick $q_0 \in Q$. There exists $k \in \mathbb{N}$ such that $q_0^k = a_1 a_2$ where $a_1, a_2 \in M$ and $a_1 \in \mathcal{A}(M)$. Since $a_i^{m(a_i)} = u(a_i) \prod_{q \in Q} q^{a_i(q)}$ for $i = 1, 2$, we set $m = m(a_1)m(a_2)$ and thus

$$q_0^{mk} = \prod_{q \in Q} q^{a_1(q) + a_2(q)},$$

and, hence, $a_1(q) = a_2(q) = 0$ for $q \in Q$, $q \neq q_0$. Therefore $a_1^{m(a_1)} = u(a_1) q_0^{a_1(q_0)}$ and by Lemma 2.2, we may replace q_0 by a_1 in Q. Replacing all elements of Q in this way, we arrive at a Cale basis of atoms.

$\square$

Throughout this section, let M be an atomic inside factorial monoid with a fixed Cale basis Q consisting of atoms; all invariants we compute are relative to this choice of Q.

Arithmetical constants like the cross number (introduced in [6]) are usually defined for the divisor class group of a monoid and integral domain, respectively. Inside factorial monoids need not possess a divisor theory, but their very structure admits imitation of those arithmetical constants. Looking at diversity in particular, we have $\operatorname{div}(x) = |Supp(x)|$, so, in a sense, diversity measures how "wide" x is (in the sense of how many distinct $q \in Q$ show up in the Cale representation of x). We will, later on, introduce the width invariant; the width of x will measure the diversity of x relative to $m(x)$ (cf. Definition 4.6).

Definition 4.2. Let M be an atomic inside factorial monoid, and fix a Cale basis Q consisting of atoms. For $x \in M^\bullet$, let $x^{m(x)} = \prod_{q \in Q} q^{x(q)}$ be the Cale representation of x by Q.

(1) We define $s(x)$ to be $s(x) = \sum_{q \in Supp(x)} x(q) = \sum_{q \in Q} x(q)$.

(2) We define the function $\varphi : M \to \mathbb{Q}_+$ by

$$\varphi(x) = \begin{cases} \frac{s(x)}{m(x)} & \text{if } x \in M^\bullet \\ 0 & \text{if } x \in M^\times \end{cases}.$$

(3) We define the **upper and lower cross numbers** of M, denoted (respectively) by $k^*(M)$ and $k_*(M)$, by

$$k^*(M) = \sup_{x \in \mathcal{A}(M)} \varphi(x) \text{ and } k_*(M) = \inf_{x \in \mathcal{A}(M)} \varphi(x).$$

(4) We define the **elasticity** of M, denoted by $\rho(M)$, to be

$$\rho(M) = \sup\left\{ \frac{r}{s} : r, s \in \mathbb{N}, x_1 x_2 \cdots x_r = y_1 y_2 \cdots y_s \text{ for } x_i, y_j \in \mathcal{A}(M) \right\}.$$

Elasticity, first introduced in the context of rings of algebraic integers by Valenza in [9], measures how "far" a given atomic monoid is from being half-factorial.[2] Recall that if M is an atomic monoid, a function $f : M \to [0, \infty)$ is called a **semi-length function** if for all $x, y \in M$:

(i) $f(xy) = f(x) + f(y)$, and
(ii) $f(x) = 0$ if and only if $x \in M^\times$.

Semi-length functions were originally introduced by Anderson and Anderson in [1] in the context of integral domains.

Proposition 4.3 ([1])**.** *Let M be an atomic monoid, and let f be a semi-length function on M. Define*

$$\alpha_f = \sup\{f(\pi) : \pi \in M \textit{ is a non-prime atom}\} \textit{ and}$$

$$\beta_f = \inf\{f(\pi) : \pi \in M \textit{ is a non-prime atom}\}$$

(and set $\alpha_f = \beta_f = 1$ if M is factorial). Then $\rho(M) \leq \frac{\alpha_f}{\beta_f}$.

Lemma 4.4. *φ is a semilength function.*

Proof. Let $x, y \in M^\bullet$. Setting $z = xy$, we have $\prod_{q \in Q} q^{z(q)m(x)m(y)} = z^{m(z)m(x)m(y)} = x^{m(z)m(x)m(y)} y^{m(z)m(x)m(y)} = \prod_{q \in Q} q^{m(z)(x(q)m(y)+y(q)m(x))}$. Hence for each $q \in Q$ we have $z(q)m(x)m(y) = m(z)(x(q)m(y) + y(q)m(x))$ which rearranges to $\frac{z(q)}{m(z)} = \frac{x(q)}{m(x)} + \frac{y(q)}{m(y)}$. Summing over all $q \in Q$ gives $\varphi(z) = \varphi(x) + \varphi(y)$. $\square$

[2]An atomic monoid M is half-factorial if every nonunit x of M has a unique length of irreducible factorization.

The following inequalities between these magnitudes will be useful later.

Lemma 4.5. *For any atomic inside factorial monoid M,*

$$\max\left\{k^*(M), \frac{1}{k_*(M)}\right\} \leq \rho(M) \leq \frac{k^*(M)}{k_*(M)}.$$

In particular, $\rho(M) = \infty$ for $k_*(M) = 0$.

Proof. Let Q be a Cale basis consisting of atoms, and let $x \in \mathcal{A}(M)$ with $x^{m(x)} = u(x)\prod_{q\in Q} q^{x(q)}$. Because Q consists of atoms, we have that $\frac{m(x)}{s(x)}, \frac{s(x)}{m(x)} \leq \rho(M)$, which implies that $\varphi(x), \varphi(x)^{-1} \leq \rho(M)$, and hence $\max\{k^*(M), k_*(M)^{-1}\} \leq \rho(M)$. Concerning the other inequality, let $x_1x_2\cdots x_r = y_1y_2\cdots y_s$ with $r, s \in \mathbb{N}$ and $x_i, y_j \in \mathcal{A}(M)$. It holds that $rk_*(M) \leq \sum\limits_{i=1}^{r}\varphi(x_i) = \sum\limits_{j=1}^{s}\varphi(y_j) \leq sk^*(M)$, and hence $\rho(M) \leq k^*(M)k_*(M)^{-1}$. □

We now analyze more closely the magnitudes $k^*(M)$ and $k_*(M)$.

Definition 4.6. Let $x \in M^\bullet$ with Cale representation $x^{m(x)} = \prod_{q\in Q} q^{x(q)}$.

(1) We define the **width** and **height** of x, denoted $w(x)$ and $h(x)$ (respectively) to be

$$w(x) = \frac{\mathrm{div}(x)}{m(x)},\; h(x) = \frac{\max\limits_{q\in Supp(x)} x(q)}{m(x)} = \frac{\max\limits_{q\in Q} x(q)}{m(x)} = \max_{q\in Q} \lambda(q, x)$$

(2) We define the **lower and upper width** of M, denoted by $w_*(M)$ and $w^*(M)$ (respectively) to be

$$w_*(M) = \inf\{w(x) : x \in \mathcal{A}(M)\} \text{ and } w^*(M) = \sup\{w(x) : x \in \mathcal{A}(M)\}.$$

(3) We define the **lower and upper height** of M, denoted by $h_*(M)$ and $h^*(M)$ (respectively) to be

$$h_*(M) = \inf\{h(x) : x \in \mathcal{A}(M)\} \text{ and } h^*(M) = \sup\{h(x) : x \in \mathcal{A}(M)\}.$$

(4) We define the **height** of M, denoted by $h(M)$, to be

$$h(M) = \sup_{x\in\mathcal{A}(M)} \max_{q\in Supp(x)} x(q) = \sup_{x\in\mathcal{A}(M), q\in Q} \lambda(q, x)$$

Lemma 4.7. *For any atomic inside factorial monoid M,*

$$\max\left\{w^*(M), h^*(M), \frac{1}{\mathrm{div}_{\mathrm{a}}(M)\cdot h_*(M)}\right\} \leq \rho(M) \leq \frac{\mathrm{div}_{\mathrm{a}}(M)\cdot h^*(M)}{\max\{w_*(M), h_*(M)\}}$$

Proof. For $\varphi(x) = \sum\limits_{q \in Supp(x)} \frac{x(q)}{m(x)}$, we have, by Theorem 2.4, that

$$\frac{\max\{x(q) : q \in Supp(x)\}}{m(x)}, \frac{\operatorname{div}(x)}{m(x)} \leq \varphi(x) \leq \operatorname{div}(x) \cdot \frac{\max\{x(q) : q \in Supp(x)\}}{m(x)},$$

and, hence $h(x), w(x) \leq \varphi(x) \leq \operatorname{div}(x) \cdot h(x)$ for all $x \in M^{\bullet}$. This implies that

$$\max\{h^*(M), w^*(M)\} \leq k^*(M) \leq \operatorname{div}_{\mathrm{a}}(M) \cdot h^*(M), \text{ and}$$
$$\max\{h_*(M), w_*(M)\} \leq k_*(M) \leq \operatorname{div}_{\mathrm{a}}(M) \cdot h_*(M).$$

Thus, by Lemma 4.5, $\rho(M) \leq k^*(M) \cdot k_*(M)^{-1} \leq \frac{\operatorname{div}_{\mathrm{a}}(M) \cdot h^*(M)}{\max\{h_*(M), w_*(M)\}}$, and $\rho(M) \geq \max\{k^*(M), k_*(M)^{-1}\} \geq \max\left\{h^*(M), w^*(M), \frac{1}{\operatorname{div}_{\mathrm{a}}(M) \cdot h_*(M)}\right\}$. □

Proposition 4.8. *Let M be an inside factorial monoid with a fixed Cale basis Q of atoms, and suppose that $m(M) = \sup\{m(x) : x \in \mathcal{A}(M)\}$ is finite. Then,*

$$\frac{\max\{div_{\mathrm{a}}(M), h(M)\}}{m(M)} \leq \rho(M) \leq m(M) \cdot div_{\mathrm{a}}(M) \cdot h(M).$$

In particular, the elasticity of M is finite if and only if both $div_{\mathrm{a}}(M)$ and $h(M)$ are finite.

Proof. For $x \in M^{\bullet}$, $\frac{\operatorname{div}(x)}{m(M)} \leq w(x) \leq \operatorname{div}(x)$, and $\frac{\max\{x(q) : q \in Supp(x)\}}{m(M)} \leq h(x) \leq \max\{x(q) : q \in Supp(x)\}$. Therefore, $\frac{\operatorname{div}_{\mathrm{a}}(M)}{m(M)} \leq w^*(M) \leq \operatorname{div}_{\mathrm{a}}(M)$, and $\frac{h(M)}{m(M)} \leq h^*(M) \leq h(M)$. Since for each $x \in M^{\bullet}$ we must have $\operatorname{div}(x) \geq 1$ and $\max\{x(q) : q \in Supp(x)\} \geq 1$, it holds that $\frac{1}{m(M)} \leq w_*(M)$ and $\frac{1}{m(M)} \leq h_*(M)$. From Lemma 4.7 we get $\frac{\max\{\operatorname{div}_{\mathrm{a}}(M), h(M)\}}{m(M)} \leq \max\{w^*(M), h^*(M)\} \leq \rho(M) \leq \operatorname{div}_{\mathrm{a}}(M) \cdot h(M) \cdot m(M)$. □

Of course, by suitably rearranging terms, the inequalities in Proposition 4.8 yield inequalities for $\operatorname{div}_{\mathrm{a}}(M)$ and $h(M)$ as well. In general, the inequalities of Proposition 4.8 will be strict. More precisely, the following characterization holds.

Corollary 4.9. *Let M be an inside factorial monoid. Then M is factorial if and only if M is atomic, $m(M)$ is finite, and both inequalities in Proposition 4.8 are equalities.*

Proof. Suppose M is factorial. By definition, we have $m(M), \operatorname{div}_{\mathrm{a}}(M), \rho(M)$, and $h(M)$ all equal to 1. Thus, the inequalities in Proposition 4.8 become equalities.

Conversely, suppose first that $\operatorname{div}_{\mathrm{a}}(M) \leq h(M)$. Then $\frac{1}{m(M)} = m(M) \cdot \operatorname{div}_{\mathrm{a}}(M)$ and hence $m(M) = 1 = \operatorname{div}_{\mathrm{a}}(M)$. The representation of an atom π by a Cale basis Q then reduces to $x = u(x)q$ for some $q \in Q$. Since M is assumed to be atomic, it follows that M is factorial.

We now assume that $\operatorname{div}_{\mathrm{a}}(M) \geq h(M)$. Then $\frac{1}{m(M)} = m(M) \cdot h(M)$ and hence $m(M) = h(M) = 1$. Again, any Cale basis Q must consist of atoms, and M must be factorial. $\square$

Proposition 4.8 implies, in particular, that $\operatorname{div}_{\mathrm{a}}(M)$ is finite if $m(M)$ and $\rho(M)$ are finite. But, as Example 5.2 in the next section shows, $\operatorname{div}_{\mathrm{a}}(M)$ may also be finite in the case where $\rho(M) = \infty$ and $m(M) < \infty$. As already mentioned in the introduction, $\operatorname{div}_{\mathrm{a}}(M)$ and $h(M)$ are, in some sense, complementary. The examples in the following section show that $\operatorname{div}_{\mathrm{a}}(M)$ may be infinite for finite $h(M)$ (Example 5.1) and vice versa (Example 5.2).

5. Examples

We close with some examples of atomic inside factorial monoids that illustrate and contrast the invariants studied in this paper. Let T denote the additive factorial monoid $\mathbb{N}_0^{\mathbb{N}}$ of sequences from $\mathbb{N}_0$ (with respect to pointwise addition) that have only finitely many nonzero entries. We discuss three examples of atomic inside factorial submonoids M of T. In particular, there will exist a fixed $n \in \mathbb{N}$ such that $nT \subseteq M$. This implies, in particular, that $Q = \{ne_i : i \in \mathbb{N}\}$ is a Cale basis of M, where e_i denotes the i^{th} unit vector of T.

Example 5.1. (*An example where $div_a(M) = \rho(M) = \infty$ and $h(M) < \infty$*)
Call $f \in T$ alternating if we have that for all $1 \leq i \leq \max\{i : f(i) > 0\}$, $f(i) > f(i+1)$ if i is odd and $f(i) < f(i+1)$ if i is even. Let A denote the additive semigroup of all such alternating functions in T, and let $F \subseteq A$ denote those alternating functions with at least two nonzero entries. Fix $n \in \mathbb{N}$, $n \geq 2$, , and let $M = (F \cup \{0\}) + nT$.

It is easily confirmed that for $k \in \mathbb{N}$, $f_k = e_1 + e_3 + e_5 + \cdots + e_{2k+1}$ is an atom of M. Obviously, $w(f_k) = \frac{\operatorname{div}(f_k)}{m(f_k)} = \frac{k+1}{n}$, and therefore $w(M) \geq \frac{k}{n}$ for all $k \in \mathbb{N}$. Therefore $w(M) = \infty$ and $\rho(M) = \infty$ by Proposition 4.8.

Now, we consider $h(M)$. For $x = \sum_{i\in\mathbb{N}} x(i)e_i \in M$, the Cale representation of x is $m(x)x = \sum y(i)(ne_i)$, where $y(i) \in \mathbb{N}_0$ is chosen so that $y(i)n = x(i)m(x)$ for each $i \in \mathbb{N}$. Therefore $h(x) = \frac{\max_{i\in\mathbb{N}} y(i)}{m(x)} = \frac{1}{n}\|x\|$, where $\|x\| = \max\{x(i) : i \in \mathbb{N}\}$ is the sup-norm on T.

Consider an atom $x = f + nt \in M$. If $f = 0$, then $t = e_i$ and hence $\|x\| = n$. If $f \in F$, then $t = 0$. For $i \in \mathbb{N}$ even, $f(i) < n$ since otherwise f could be written as a sum of ne_i and $f' \in F$. For $i \in \mathbb{N}$ odd, suppose that $f(i) > n + f(i+1)$ and $f(i) > n + f(i-1)$. Then f may again be written as a sum of ne_i and some $f' \in F$. Therefore, for i odd, we must have $f(i) \leq n + f(i+1) < n + n$ or $f(i) \leq n + f(i-1) < n + n$, and in any case, $f(i) < 2n$ for i odd. This shows that $\|x\| \leq 2n$, and hence for $x \in \mathcal{A}(M)$, $h(x) \leq 2$, and hence $h(M) \leq 2$.

Example 5.2. (*An example where* $div_a(M) < \infty$ *and* $h(M) = \rho(M) = \infty$)
For $k \in \mathbb{N}$, let $f_k = ke_k$. Let F denote the additive semigroup generated by the f_k. Fix $n \geq 2$, and consider the submonoid of T given by $M = (F \cup \{0\}) + nT$. It is straightforward to show that $\mathcal{A}(M) = \{f_k : k \in \mathbb{N}, n \nmid k\} \cup \{ne_k : k \in \mathbb{N}, k \nmid n\} \cup \{ne_n\}$. A Cale representation for f_k is $nf_k = k(ne_k)$. It follows that $m(f_k) = \frac{n}{gcd(k,n)}$ and $f_k(ne_k) = \frac{k}{gcd(k,n)}$. Therefore, $h(f_k) = \frac{f_k(ne_k)}{m(f_k)} = \frac{k}{n}$, implying that $h(M) = \infty$. By Proposition 4.8, it follows that $\rho(M) = \infty$.

We now turn our attention to $\mathrm{div_a}(M)$. From our work above, we have $w(f_k) = \frac{\mathrm{div}(f_k)}{m(f_k)} = \frac{1}{n/\gcd(k,n)} \leq 1$. Furthermore for all $k \nmid n$, $w(ne_k) = w(ne_n) = 1$. Therefore $\mathrm{div_a}(M) = 1$ and, in fact, every atom of M is almost primary.

Example 5.3. (*An example where* $div_a(M) = h(M) = \rho(M) = \infty$)
For integer $k \geq 2$, let $f_k = (k, k-1, k-2, \cdots, 2, 1, 0, 0, 0, \cdots)$, and let F be the additive semigroup generated by the f_k. Fix $n \geq 2$, and let $M = (F \cup \{0\}) + nT$. We see that $\mathcal{A}(M) = \{f_k : k \in \mathbb{N}\} \cup \{ne_k : k \in \mathbb{N}\}$. The atoms f_k have Cale representation $nf_k = k(ne_1) + (k-1)(ne_2) + \cdots + 2(ne_{k-1}) + (ne_k)$. It then follows that $m(f_k) = n$. Therefore $w(f_k) = \frac{\mathrm{div}(f_k)}{m(f_k)} = \frac{k}{n}$, and hence $\mathrm{div_a}(M) = \infty$. Similarly, $h(f_k) = \frac{\max\{f_k(ne_i) : i \in \mathbb{N}\}}{m(f_k)} = \frac{k}{n}$, yielding $h(M) = \infty$. Finally, $\rho(M) = \infty$ by Proposition 4.8.

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