Fibonacci Nim and a Full Characterization of Winning Moves

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Abstract

In this paper we will fully characterize all types of winning moves in the "take-away" game of Fibonacci Nim. We prove the known winning algorithm as a corollary of the general winning algorithm and then show that no other winning algorithms exist. As a by-product of our investigation of the game, we will develop useful properties of Fibonacci numbers. We conclude with an exploration of the probability that unskilled player may beat a skilled player and show that as the number of tokens increase, this probability goes to zero exponentially.

1 Introduction

We begin with a brief introduction to the idea of "take-away" games. Schwenk defined "take-away" games to be a two-person game in which the players alternately diminish an original stock of tokens subject to various restrictions, with the player who removes the last token being the winner [5].

In the generalized *take-away* game, $\tau(k) = \eta(k-1) - \eta(k)$ where $\eta(k)$ is the number of tokens remaining after the k^{th} turn so that $\tau(k)$ is the number of tokens removed on the k^{th} turn. Additionally, for all $k \in \mathbb{N}$, $k \neq 1$, we have $\tau(k) \leq m_k$, where m_k is some function of $\tau(k-1)$. Specifically in Fibonacci Nim, we have $m_k = 2\tau(k)$ for $k > 1$. We will immediately move away from this notation and develop additional notation as it is required. We provide a simple example to familiarize the reader with the game.

Example 1. Let $n = 10$. Player one may remove 1 through 9 tokens. Suppose player one removes 3 tokens. Then, player two may now remove 1 through 2(3)=6 tokens. Play continues until one of the players removes the last token.

We will rely heavily on results from the original Lekkerkerker paper [4], specifically the *Zeckendorff Representation* of natural numbers as a sum of Fibonacci Numbers.

The Fibonacci numbers are the positive integers generated by the recursion $F_k = F_{k-1} + F_{k-2}$, where $F_1 = 1 = F_2$ and $k \in \mathbb{N}$. Let $F = \{F_2, F_3, ...F_k, ...\} = \{1, 2, 3, 5, ...\}$. This is the subset of Fibonacci numbers we will reference throughout this paper. We now present The Zeckendorf Representation theorem without proof. A proof of this theorem may be found in [2].

Theorem 1 (Zeckendorff Representation Theorem)**.** *Let $n \in \mathbb{N}$. For $i, r \in \mathbb{N}$ we have $n = F_{i_r} + F_{i_{r-1}} + \cdots + F_{i_1}$ where $i_r - (r-1) > i_{r-1} - (r-2) > \cdots > i_2 - 1 > i_1 \geq 2$. Further, this representation is unique.*

The Zeckendorff Representation Theorem states that every positive integer can be written as a sum of non-consecutive Fibonacci numbers from the set F and that this representation is unique. It follows that $F_{i_r} > F_{i_{r-1}} > ... > F_{i_1}$. We will refer to the Zeckendorff Representation theorem frequently, therefore we abbreviate this by ZRT.

Example 2. $12 = (1)F_6 + (0)F_5 + (1)F_4 + (0)F_3 + (1)F_2 = 8 + 3 + 1$.

Corollary 1. *If $F_{k+1} > n \geq F_k$, then F_k is the largest number in the Zeckendorff representation of n.*

Proof. If $F_{k+1} > n \geq F_k$ then by Zeckendorff's theorem we can write $(n - F_k) = F_d + \cdots + F_{i_1}$. We claim $k - 1 > d$. Suppose not, then $d \geq (k-1)$, thus $n = F_k + F_d + \cdots + F_{i_1} \geq F_k + F_d \geq F_k + F_{k-1} = F_{k+1}$. However, $F_{k+1} > n \geq F_{k+1}$ is a contradiction. Thus, $k - 1 > d$ so that $n = F_k + F_d + \cdots + F_{i_1}$ is a valid and thus the only representation of n by ZRT. □

The corollary above shows that for any $n \in \mathbb{N}$ where $F_{k+1} > n \geq F_k$, the Zeckendorff Representation of n must contain F_k. Therefore, we iteratively may take the maximal Fibonacci number less than n, say F_k, subtract it from n which yields $n - F_k = n' = F_{i_{r'}} + F_{i_{r-1}'} + \cdots + F_{i_{1'}}$ and repeat this process to find each Fibonacci number in the representation of the original number, n.

Definition 1. Let $n = F_{i_r} + F_{i_{r-1}} + \cdots + F_{i_1}$ where $r, i, n \in \mathbb{N}$. We define $T(n) = F_{i_1}$. That is, $T(n)$ is the smallest number in the Zeckendorff representation.

Definition 2. Let $n = F_{i_r} + F_{i_{r-1}} + \cdots + F_{i_1}$ where $r, i, n, j \in \mathbb{N}$. We now define the *length j tail* to be the specific sum of j consecutive* Fibonacci numbers in the Zeckendorff representation of n beginning with the smallest number, F_{i_1}. We set $T_1(n) = T(n)$ for consistency. Then, $T_j(n) = T(n) + T_{i-1}(n - T(n))$.

The *consecutive** in definition (2) refers to the the subscripts i_j, i_{j+1} for some $j \in \mathbb{N}$. By the above definitions, we see that the *length j tail* of n is $T_j(n) = F_{i_j} + F_{i_{j-1}} + \cdots + F_{i_1}$ where $r \geq j \geq 1$.

Example 3. Consider $33 = F_8 + F_6 + F_4 + F_2 = 21 + 8 + 3 + 1$ and $12 = F_6 + F_4 + F_2 = 8 + 3 + 1$. Then, the *length 3 tail* $T_3(33) = F_6 + F_4 + F_2 = 8 + 3 + 1$ and $T_3(12) = F_6 + F_4 + F_2 = 8 + 3 + 1$. Hence, 33 and 12 have the same length 3 tail.

Remark 1. By the definition of a length j tail, if $T_j(n) = T_j(m)$, then for any $j \geq s \geq 1$, we have $T_s(n) = T_s(m)$.

Let $n = F_{i_r} + F_{i_{r-1}} + \cdots + F_{i_1}$ be the Zeckendorff representation where $F_{i_r} > F_{i_{r-1}} > \cdots > F_{i_1}$. Suppose there are n tokens in the pile during the current turn. The known winning algorithm for Fibonacci Nim has the current player take the length-1 tail of n. That is, the player removes $T(n) = F_{i_1}$ tokens. We will prove that this is a winning algorithm in the next section.

In what follows, we will extend the known winning algorithm to include tails that satisfy certain criteria for some given n. We then will prove that this is a complete collection of winning moves and that no others exist. We end this paper by introducing a *losing position strategy* and then derive an upperbound on the probability that an unskilled player may beat a skilled player.

2 Fibonacci Nim Strategy

We begin this section by discussing how to win Fibonacci Nim. In the remainder of this paper, we use $n = F_{i_r} + F_{i_{r-1}} + \cdots + F_{i_1}$ with $n, r, i \in \mathbb{N}$ as the Zeckendorff representation for some n.

Assume there are n tokens in a given turn which the player whose turn it is may remove from. Let $2p$ denote the maximum number of tokens this player may remove from the n tokens. We can denote this postion by $(n, 2p)$. Note, this implies that the previous player removed precisely p tokens.

Definition 3. A *losing position* is such that given the position $(n, 2p)$, $T(n) > 2p$. A *winning position* is any non-losing position. A *winning move* is such that it results in the next position being a losing position. A *losing move* is any non-winning move.

We see by definition (3) that we always want to leave our opponent in a losing position where $T(n) > 2p$. That is, a position where our opponent cannot remove any length j tail, $T_j(n)$. As an immediate consequence, if our opponent cannot remove a tail of n, certainly he cannot remove all of n to win since $n \geq T_j(n) \geq T(n) > 2p$. Therefore, if we can successively give our opponent a losing position, we can ensure we win.

Lemma 1. *For every $i \in \mathbb{N}$ where $i \geq 3$, $2F_{i-1} \geq F_i$ and $F_{i+1} > 2F_{i-1}$*

Proof. We have $2(F_2) = 2(1) = 2 = F_3$ and $F_4 = 3 > 2 = 2(1) = 2(F_2)$. Assume $2F_{i-1} \geq F_i$ and $F_{i+1} > 2F_{i-1}$. We have $2(F_i) = 2(F_{i-1} + F_{i-2}) \geq 2F_{i-1} + F_{i-2} = F_i + F_{i-1} = F_{i+1}$ since for each for $j \in \mathbb{N}$, $F_j \geq 1$. Similarly, $F_{i+2} = F_{i+1} + F_i = (F_i + F_{i-1}) + (F_{i-1} + F_{i-2}) > F_i + (F_{i-1} + F_{i-2}) = 2F_i$. □

Lemma (2) below implies that if on a given turn our opponent has a losing position to play from, regardless of how he plays, our next play will be from a a winning position.

Lemma 2. *Let $n \in \mathbb{N}$. For any p with $T(n) > p$, $(n - p, 2p)$ is a winning position.*

Proof. Let $n \in \mathbb{N}$. Assume $T(n) > p$. We have, $n - p = F_{i_r} + \cdots + F_{i_1} - p$. Define $m = T(n) - p = F_{i'_r} + \cdots + F_{i'_1}$. Suppose $(n-p, 2p)$ is a losing position. Then, $T(n-p) > 2p$ and by lemma (1), $2F_{i'_1 - 1} \geq F_{i'_1} > 2p$. Hence, the Zeckendorff representation of p does not include $F_{i'_1 - 1}$, thus $p = F_{i''_r} + ... + F_{i''_1}$ where $F_{i'_1 - 1} > F_{i''_r}$. But then, $n = F_{i_r} + \cdots + F_{i_2} + F_{i'_r} + \cdots + F_{i'_1} + F_{i''_r} + \cdots + F_{i''_1}$ is a valid Zeckendorff representation of n. This is a contradiction since Zeckendorff representations are unique. Hence, we must have $2p \geq T(n-p)$. Since $F_{i_2} > T(n) > T(n) - p$, then, $n - p = F_{i_r} + \cdots + F_{i_2} + m$ is a valid Zeckendorff representation of $n - p$ and hence the only representation. Thus, the next position, $(n - p, 2p)$ has $2p \geq T(n - p)$ so that $(n - p, 2p)$ is a winning position. □

Lemma (3) below paired with Lemma (2) proves the known winning strategy. That is, if we take the length one tail of n, $T(n)$, the next position is a losing position. Successively implementing this lemma results in winning the game in a finite number of moves.

Lemma 3. *Let $n \in \mathbb{N}$. Set $p = T(n)$. Then $(n - p, 2p)$ is a losing position.*

Proof. Let $n \in \mathbb{N}$. Set $p = T(n)$. Suppose for some $k \in \mathbb{N}$, $F_k = T(n) = F_{i_1}$. By theorem (1), $F_{i_2} \geq F_{k+2}$. Then, by lemma (1), $F_{i_2} \geq F_{k+2} > 2F_k = 2p$. By uniqueness of ZRT, $n - p = F_{i_r} + \cdots + F_{i_2}$ and $(n-p, 2p)$ has $T(n-p) = F_{i_2} > 2p$. Hence, $(n-p, 2p)$ is a losing position. □

For now, we state that not every tail may always be taken from n to produce a losing position. In the following subsections, we will prove this rigorously and derive results which show exactly which tails may be removed to put our opponent in a losing position. Theorem (2) is this section's main result. Namely, it proves that removing length j tails of n are the *only* winning moves for $n \in \mathbb{N}$.

Theorem 2 (Fundamental Theorem of Fibonacci Nim)**.** *Let $n \in \mathbb{N}$. For any $p \notin \{T_{r-1}(n), T_{r-2}(n), ..., T(n)\}$, $(n-p, 2p)$ is a winning position.*

Proof. Let $n \in \mathbb{N}$ and suppose our opponent has removed p tokens. Then the current position is $(n-p, 2p)$. Assume $T(n-p) > 2p$, that is, $(n-p, 2p)$ is a losing position. If $p = T_j(n)$ for some $r > j \geq 1$, then $p \in \{T_{r-1}(n), T_{r-2}(n), ..., T(n)\}$. This leaves two cases to examine: (1) p is a 'sum' of terms F_{i_t} where $r \geq t \geq 1$ and $p \neq T_j(n)$ for some $r > j \geq 1$ or (2) $p \neq T_j(n)$ for some $r > j \geq 1$ and p is not of the form given in case (1).

Case 1: Our opponent removes $p = a_r F_{i_r} + a_{r-1} F_{i_{r-1}} + \cdots + a_1 F_{i_1}$ where each $a_j \in \{0, 1\}$ for $j \in [1, i_r]$ and there exists at least one pair (a_j, a_{j+1}) such that $a_j = 0$ and $a_{j+1} = 1$ in the representation of p. Then, $p \neq T_j(n)$ for some $r > j \geq 1$. WLOG, let (a_j, a_{j+1}) be the minimal pair such that $a_j = 0$ and $a_{j+1} = 1$ in the representation of p. Define $n' = (F_{i_r} - a_r F_{i_r}) + \cdots + (F_{i_{j+1}} - a_{j+1} F_{i_{j+1}}) + (F_{i_{j-1}} - a_{j-1} F_{i_{j-1}}) + \cdots + (F_{i_1} - a_1 F_{i_1})$. Then, $n - p = F_{i_r} + F_{i_{r-1}} + \cdots + F_{i_1} - (a_r F_{i_r} + a_{r-1} F_{i_{r-1}} + \cdots + a_1 F_{i_1}) = n' + F_{i_j}$ which is a valid Zeckendorff representation and hence the only representation of $n - p$. Since (a_j, a_{j+1}) is minimal, $T(n-p) = F_{i_j}$. We have, $2p > F_{i_{j+1}} > T(n-p)$, thus $(n-p, 2p)$ is a winning position and we have reached a contradiction.

Case 2: Our opponent removes p tokens such that $p \neq a_r F_{i_r} + a_{r-1} F_{i_{r-1}} + ... + a_1 F_{i_1}$ where each $a_j \in \{0, 1\}$. Since $(n-p, 2p)$ is a losing position, by lemma (2) we must have $p > T(n)$. WLOG, let $T_j(n)$ for $r > j \geq 1$ be the minimal tail such that $p > T_j(n)$. By assumption, $p \neq T_j(n)$. We have $F_{i_{j+1}} + T_j(n) > p > T_j(n)$ so that $F_{i_{j+1}} > p - T_j(n) > 0$. Define $\delta p = p - T_j(n)$ so that $p = T_j(n) + \delta p$. Let $m = n - T_j(n)$. Then, $n - p = m + T_j(n) - (T_j(n) + \delta p) = m - \delta p$. Since $T(m) > \delta p$, by lemma (2) and the uniqueness of Zeckendorff representations, $(m - \delta p, 2\delta p)$ is a winning position. It follows that $2p > 2\delta p \geq T(m - \delta p) = T(n-p)$. Therefore, $(n-p, 2p)$ is a winning position and we have reached a contradiction.

Hence, removing some $p \neq T_j(n)$ for some $r > j \geq 1$ results in a winning position. Since there is only one other possible move, removing some tail $F_j(n)$, it follows that if $(n-p, 2p)$ is a losing move, then $p = T_j(n)$ for some $r > j \geq 1$. □

Remark 2. By definition (3) and theorem (2), removing $T_j(n)$ tokens where $r > j \geq 1$ will force an immediate losing position to our opponent when $F_{i_{j+1}} > 2T_j(n)$.

In section (2) we have shown that the only possible winning moves in Fibonacci Nim are those that are partial consecutive* sums or, tails of the Zeckendorff representation of the number of tokens in that turn. In the next section, we determine which tails force losing positions and how to identify these tails based soley on the Zeckendorff representation for a given n.

3 Winning Tails

In this Section, we will show how to take the result from remark (2): *removing* $T_j(n)$ *tokens where* $r > j \geq 1$ *will force an immediate losing position to our opponent when* $F_{i_{j+1}} > 2T_j(n)$ and identify which tails satisfy this condition. Existence of winning moves was proved for *Dynamic One-Pile Nim* in a paper by Holshouser, Reiter and Rudzinski [3]; Fibonacci Nim is classified as a dynamic one-pile nim game in their paper. Below, we validate the existence of these moves as well as carefully show exactly how to find these winning moves. In addition, we have included a table at the end of this paper to present these results for the first 90 positive integers.

We are concerned with which tails can be taken and which cannot. That is, if $n = F_{i_r} + F_{i_{r-1}} + \cdots + F_{i_1}$, when is $F_{i_{j+1}} > 2T_j(n)$ for $r > j \geq 1$? We accomplish this by looking at an arbitrary tail of n, $T_j(n)$. We classify exactly when taking $T_j(n)$ results in leaving a losing position to our opponent.

We begin by setting $a_{j+1} = i_{j+1} - i_j$ and $a_j = i_j - i_{j-1}$. Then, a_{j+1} and a_j are the differences in the subscripts of consecutive* Fibonacci numbers in a Zeckendorff representation of n. In this section we will show that for any F_{i_j}, by considering the 'gaps' around it, where the gaps are the differences above, we can determine if removing $T_{i_j}(n)$ tokens give our opponent a losing position. For us to do this, we must first introduce the *gap-vector*.

Definition 4. Let $n = F_{i_r} + F_{i_{r-1}} + \cdots + F_{i_1}$. We define the *gap-vector* of n to be $G(n) = (a_r, a_{r-1}, ..., a_2; a_1)$ where $a_r = i_r - i_{r-1}$, $a_{r-1} = i_{r-1} - i_{r-2}$, ... ,$a_2 = i_2 - i_1$ and $a_1 = i_1$. We also define $|G(n)| = r$, where r is the number of summands in the Zeckendorf representation of n.

Example 4. Let $n = 129 = F_{11}+F_9+F_5+F_2$. Then, $G(129) = (11-9, 9-5, 5-2; 2) = (2, 4, 3; 2)$ and $|G(129)| = 4$.

The *gap-vector* of n shows the difference of the subscripts of the consecutive* Fibonacci numbers in the Zeckendorff representation of n (again, consecutive* refers to the the subscripts i_j, i_{j+1} for some $j \in \mathbb{N}$). The last coordinate of the *gap-vector* is the subscript of the smallest Fibonacci number present in the Zeckendorff representation of n. It follows, that we can reconstruct n by using the *gap-vector* of n.

Example 5. Let $G(n) = (2, 4, 3; 2)$. Then, F_2 is the first Fibonacci number in the representation of n. From here, we can build the rest of the numbers: 2+3 = 5, so F_5 is the next number; 4+5 = 9, so F_9 is the third number and 9+2=11, so F_{11} is the last number in the representation of n. Hence, $n = F_{11} + F_9 + F_5 + F_2 = 129$.

It is worth mentioning that by ZRT each $a_j \geq 2$ for $j \in \mathbb{N}$. We now begin to examine which tails provide winning moves. Consider $p = T_j(n)$ for some $n, j \in \mathbb{N}$. We will classify exactly when $T_j(n)$ is a winning move and hence leaves the opponent the losing position $(n - p, 2p)$.

Notational remark: For the following lemmas, we introduce the symbol $(k : 2)$ such that $(k : 2) \in \{2, 3\}$ where $(k : 2) \equiv k \bmod 2$. Similarly, $(k : 3) \in \{2, 3, 4\}$ where $(k : 3) \equiv k \bmod 3$. For example, $F_8 + ... + F_{k:2} = F_8 + ... + F_2$ since $(k : 2) \equiv 8 \bmod 2$ and $(k : 2) \in \{2, 3\}$.

For the remainder of this section, we will give a lemma and then a corollary. The lemma provides properties of particular Fibonacci series. The corollaries tie the lemma into Fibonacci Nim.

Lemma 4. *For $k \geq 5$, $F_k > 2(F_{k-3} + F_{k-5} + \cdots + F_{k:2})$.*

Proof. Let $k = 5$, then $F_5 = 5 > 2(1) = 2(F_2)$. Let $k = 6$, then $F_6 = 8 > 4 = 2(2) = 2(F_3)$. Suppose $F_k > 2(F_{k-3} + F_{k-5} + \cdots + F_{k:2})$. Then by induction hypothesis, $2F_{k-1} + F_k > 2F_{k-1} + 2(F_{k-3} + F_{k-5} + \cdots + F_{k:2}) = 2(F_{k-1} + ... + F_{k:2})$. But, $F_{k+2} = F_{k+1} + F_k > 2F_{k-1} + F_k$ by lemma (1). Hence, $F_{k+2} > 2(F_{k-1} + \cdots + F_{k:2})$. $\square$

Corollary 2. *Let $G(n) = (a_r, a_{r-1}, ..., a_2; a_1)$ where $r > 1$ and $a_j \geq 2$ for $r \geq j > 1$. If $a_{q+1} \geq 3$ for some $r > q > 1$, then for $p = T_q(n)$, $(n - p, 2p)$ is a losing position.*

Proof. Let $G(n) = (a_r, a_{r-1}, ..., a_2; a_1)$ where $r > 1$ and $a_j \geq 2$ for $r \geq j > 1$. Suppose $a_{q+1} \geq 3$ for some $r > q > 1$ and set $p = T_q(n)$. Then $i_{q+1} \geq i_q + 3$. By lemma (4), we have $F_{i_{q+1}} > 2(F_{i_q} + \cdots + F_{i_1}) = 2T_q(n)$. We have, $n - p = F_{i_r} + ... + F_{i_{q+1}}$ by the uniqueness of Zeckendorff representations, hence $T(n - p) = F_{i_{q+1}} > 2T_q(n) = 2p$ and $(n - p, 2p)$ is a losing position. $\square$

We see by the above corollary that if $G(n) = (a_r, ..., a_2; a_1)$ contains coordinates $a_j \geq 2$ and some $a_{q+1} \geq 3$ we can always remove the tail beginning with the Fibonacci number F_{i_q}. But notice, by ZRT, every representation will have $a_j \geq 2$ for $r \geq j \geq 2$. Hence, we have just shown by corollary (2) that given some $n = F_{i_r} + \cdots + F_{i_{j+1}} + F_{i_j} + \cdots + F_{i_1}$, if $i_{j+1} - 3 \geq i_j$, then removing $p = T_j(n)$ results in $(n - p, 2p)$ being a *losing position*. Therefore it follows that we need only to consider when $i_{j+1} - 2 = i_j$ to classify the remainder of winning tails.

Lemma 5. *For $k \geq 8$, $F_k > 2(F_{k-2} + F_{k-6} + F_{k-8} + \cdots + F_{k:2})$.*

Proof. Let $k = 8$, then $F_8 = 21 > 2(8 + 1) = 2(F_6 + F_2)$. Let $k = 9$, then $F_9 = 34 > 30 = 2(13 + 2) = 2(F_7 + F_3)$. Assume $F_k > 2(F_{k-2} + F_{k-6} + F_{k-8} + \cdots + F_{k:2})$. By induction hypothesis we have, $F_{k+2} = F_{k+1} + F_k > F_{k+1} + 2F_{k-2} + 2(F_{k-6} + \cdots + F_{k:2})$. But, $F_{k+1} + 2F_{k-2} = F_k + F_{k-1} + 2F_{k-3} + 2F_{k-4}$. By lemma (1), $2F_{k-3} > F_{k-2}$. Hence, $F_{k+1} + 2F_{k-2} > F_k + F_{k-1} + F_{k-2} + 2F_{k-4} = 2(F_k + F_{k-4})$. Then, $F_{k+2} > 2(F_k + F_{k-4} + F_{k-6} + \cdots + F_{k:2})$. $\square$

Corollary 3. *Let $G(n) = (a_r, a_{r-1}, ..., a_2; a_1)$ where $r > 2$ and $a_j \geq 2$ for $r \geq j > 1$. If $a_q \geq 4$ and $a_{q+1} = 2$ for some $r \geq q > 1$, then for $p = T_q(n)$, $(n - p, 2p)$ is a losing position.*

Proof. Let $G(n) = (a_r, a_{r-1}, ..., a_2; a_1)$ where $r > 1$ and $a_j \geq 2$ for $r \geq j > 1$. Suppose $a_q \geq 4$ for some $r \geq q > 1$ and set $p = T_q(n)$. Then $i_{q+1} - 2 = i_q \geq i_{q-1} + 4$. By lemma (5), we have $F_{i_{q+1}} > 2(F_{i_q} + \cdots + F_{i_1}) = 2T_q(n)$. We have, $n - p = F_{i_r} + ... + F_{i_{q+1}}$ by the uniqueness of Zeckendorff representations, hence $T(n - p) = F_{i_{q+1}} > 2T_q(n) = 2p$ and $(n - p, 2p)$ is a losing position. $\square$

We see by corollary (3) that if $G(n) = (a_r, ..., a_2; a_1)$ contains coordinates $a_j \geq 2$ and some $a_q \geq 4$ and $a_{q+1} = 2$, we can always remove the tail beginning with the Fibonacci number F_{i_q}. Hence, we have just shown that given some $n = F_{i_r} + \cdots + F_{i_{j+1}} + F_{i_j} + \cdots + F_{i_1}$, if $i_{q+1} - 2 = i_q \geq i_{q-1} + 4$, then removing $p = T_j(n)$ results in $(n - p, 2p)$ being a *losing position*. By corollaries (2) and (3), we have just shown that if we have $a_{q+1} \geq 3$ or, if $a_{q+1} = 2$ and $a_q \geq 4$, then $p = T_q(n)$ is a winning move, that is, $(n - p, 2p)$ is a losing position. Thus, what remains to examine are the cases $a_{q+1} = 2 = a_q$ and $a_{q+1} = 2$ and $a_q = 3$. We begin with the former.

Lemma 6. *For $k \geq 6$, $F_k \leq 2(F_{k-2} + F_{k-4})$*

Proof. Let $k = 6$. Then, $F_6 = 8 = 2(3+1) = 2(F_4 + F_2)$. For any $k > 6$, we have $F_k = 2F_{k-2} + F_{k-3}$. By lemma (2), $2F_{k-4} \geq F_{k-3}$. Hence, $F_k = 2F_{k-2} + F_{k-3} \leq 2(F_{k-2} + F_{k-4})$. $\square$

Corollary 4. *Let $G(n) = (a_r, a_{r-1}, ..., a_2; a_1)$ where $r > 2$ and $a_j \geq 2$ for $r \geq j > 1$. If $a_{q+1} = 2 = a_q$ for some $r \geq q > 1$, then for $p = T_q(n)$, $(n-p, 2p)$ is a winning position.*

Proof. Let $G(n) = (a_r, a_{r-1}, ..., a_2; a_1)$ where $r > 1$ and $a_j \geq 2$ for $r \geq j > 1$. Suppose $a_{q+1} = 2 = a_q$ for some $r \geq q > 1$ and set $p = T_q(n)$. Then $i_{q+1} - 2 = i_q = i_{q-1} + 2$. By lemma (6), we have $F_{i_{q+1}} \leq 2(F_{i_q} + F_{i_{q-1}}) \leq 2(F_{i_q} + \cdots + F_{i_1}) = 2T_q(n)$. We have, $n - p = F_{i_r} + ... + F_{i_{q+1}}$ by the uniqueness of Zeckendorff representations, but $T(n-p) = F_{i_{q+1}} \leq 2T_q(n) = 2p$. Thus, $(n-p, 2p)$ is a winning position. $\square$

We are now left with the case $a_{q+1} = 2$ and $a_q = 3$. It turns out, this case is slightly more complicated than the previous cases. We will show that given $G(n) = (a_r, a_{r-1}, ..., a_2; a_1)$ where $r > 1$ and $a_j \geq 2$ for $r \geq j > 1$, if there exists some q such that every $a_k \geq 3$ for $r > q \geq k > 1$, then $T_q(n)$ for $r > q > 1$ is a winning move. If however, we have some $a_k = 2$ for $q \geq k > 1$, then $T_q(n)$ for $r > q > 1$ is a losing move. We begin with the former.

Lemma 7. *For $k \geq 10$, $F_k - 2(F_{k-2} + F_{k-5} + F_{k-8} + ... + F_{k:3}) = q$ where $q \in \{1, 2\}$.*

Proof. We prove the lemma in cases for $F_{k:3}$. Specifically for some $m \in \mathbb{N}$ and $m \geq 3$, $F_{k:3} = F_2$ when $k = 3m+1$ since $3m + 1 - (2 + 3(m-1)) = 2$ and $F_{k:3} = F_3$ when $k = 3m+2$ since $3m + 2 - (2 + 3(m-1)) = 3$. $F_{k:3} = F_4$ when $k = 3m$ since $3m - (2 + 3(m-2)) = 4$. Note, if we have $3m - (2 + 3(m-1)) = 1$, we will not have a valid Zeckendorff representation, hence we must reduce our multiple by one, which yields $3(m-2)$ above.

Case 1: Let $F_{k:3} = F_2$ and let $m = 3$ so that $k = 3m+1 = 10$. Then $F_{10} - 2(F_8 + F_5 + F_2) = 55 - 2(21 + 5 + 1) = 1$. Let $m > 3$ so that $k > 10$ and assume $F_{3m+1} - 2(F_{3m-1} + F_{3m-4} + F_{3m-7} + ... + F_5 + F_2) = 1$. Then, $F_{3m+1} + 2F_{3m+2} - 2F_{3m+2} - 2(F_{3m-1} + F_{3m-4} + ... + F_5 + F_2) = 1$ by inductive hypothesis. But, $F_{3(m+1)+1} = F_{3m+4} = F_{3m+3} + F_{3m+2} = 2F_{3m+2} + F_{3m+1}$. Hence, $F_{3m+4} - 2(F_{3m+2} + F_{3m-1} + F_{3m-4} + ... + F_5 + F_2) = 1$.

Case 2: Now let $F_{k:3} = F_3$ and let $m = 3$ so that $k = 11$. Then, $F_{11} - 2(F_9 + F_6 + F_3) = 89 - 2(34 + 8 + 2) = 1$. Let $m > 3$ so that $k > 11$ and assume $F_{3m+2} - 2(F_{3m} + F_{3m-3} + F_{3m-6} + ... + F_6 + F_3) = 1$. Then $F_{3m+2} + 2F_{3m+3} - 2F_{3m+3} - 2(F_{3m} + F_{3m-3} + ... + F_6 + F_3) = 1$ by inductive hypothesis. But, $F_{3(m+1)+2} = F_{3m+5} = F_{3m+4} + F_{3m+3} = 2F_{3m+3} + F_{3m+2}$. Hence, $F_{3m+5} - 2(F_{3m+3} + F_{3m} + F_{3m-3} + ... + F_6 + F_3) = 1$.

Case 3: Finally, let $F_{k:3} = F_4$ and let $m = 4$ so that $k = 12$. Then, $F_{12} - 2(F_{10} + F_7 + F_4) = 144 - 2(55 + 13 + 3) = 2$. Let $m > 4$ so that $k > 12$ and assume that $F_{3m} - 2(F_{3m-2} + F_{3m-5} + F_{3m-8} + ... + F_7 + F_4) = 2$. Then, $F_{3m} + 2F_{3m+1} - 2F_{3m+1} - 2(F_{3m-2} + F_{3m-5} + ... + F_7 + F_4) = 2$ by inductive hypothesis. But, $F_{3(m+1)} = F_{3m+3} = F_{3m+2} + F_{3m+1} = 2F_{3m+1} + F_{3m}$. Hence, $F_{3m+3} - 2(F_{3m+1} + F_{3m-2} + F_{3m-5} + ... + F_7 + F_4) = 2$.

Hence, in each case we find that $F_k - 2(F_{k-2} + F_{k-5} + F_{k-8} + ... + F_{k:3}) = q$ with $q \in \{1, 2\}$. $\square$

Remark 3. It should be clear from lemma (7) that for $k \geq 10$, we have $F_k > 2((F_{k-2} + F_{k-5} + F_{k-8} + ... + F_{k:3})$.

Corollary 5. *Let $G(n) = (a_r, a_{r-1}, ..., a_2; a_1)$ where $r > 2$ and $a_j \geq 2$ for $r \geq j > 1$. If $a_{q+1} = 2$ and $a_j \geq 3$ for $q \geq j \geq 1$, then for $p = T_q(n)$, $(n-p, 2p)$ is a losing position.*

Proof. Let $G(n) = (a_r, a_{r-1}, ..., a_2; a_1)$ where $r > 2$ and $a_j \geq 2$ for $r \geq j > 1$. Suppose $a_{q+1} = 2$ and $a_j \geq 3$ for $q \geq j \geq 1$ and set $p = T_q(n)$. Then $i_{q+1} - 2 = i_q$ and $i_{j+1} - 3 \geq i_j$ for every $q > j \geq 1$. By lemma (7) and remark (3), we have $F_{i_{q+1}} > 2(F_{i_q} + \cdots + F_{i_1}) = 2T_q(n)$. We have, $n - p = F_{i_r} + ... + F_{i_{q+1}}$ by the uniqueness of Zeckendorff representations and $T(n-p) = F_{i_{q+1}} > 2T_q(n) = 2p$. Thus, $(n-p, 2p)$ is a losing position. □

By corollary (5), if $G(n) = (a_r, ..., a_2; a_1)$ contains coordinates $a_j \geq 2$ and if for some $a_{q+1} = 2$ we have for every $q \geq k \geq 1$, $a_k \geq 3$ then we may remove the tail beggining with the Fibonacci number F_{i_q}, that is, $T_q(n)$. All that remains to show is the case when at least one $a_k = 2$.

Lemma 8. *For $k \geq 6$, $F_k - (F_{k-1} + F_{k-4} + F_{k-7} + \cdots + F_{k:3}) > 1$.*

Proof. We prove the lemma in cases for $F_{k:3}$. Specifically for some $m \in \mathbb{N}$ and $m \geq 2$, $F_{k:3} = F_2$ when $k = 3m$ since $3m - (1 + 3(m-1)) = 2$ and $F_{k:3} = F_3$ when $k = 3m+1$ since $3m + 2 - (1 + 3(m-1)) = 3$. $F_{k:3} = F_4$ when $k = 3m+2$ since $3m + 2 - (1 + 3(m-1)) = 4$.

Case 1: Let $m = 2$ so that $k = 6$. Then, $F_6 - (F_5 + F_2) = 8 - (5+1) = 2$. Assume $F_k - (F_{k-1} + F_{k-4} + F_{k-7} + \cdots + F_{k:3}) > 1$ for $m > 2$. Then, by induction hypothesis, we have $F_{3m} + F_{3m+2} - F_{3m+2} - (F_{3m-1} + F_{3m-4} + \cdots + F_2) > 1$. But, $F_{3m+3} = F_{3m+2} + F_{3m+1} > F_{3m+2} + F_{3m}$ and $F_{3m+1} - F_{3m} > 2$ when $m > 2$ by construction. Hence, $F_{3m+3} - (F_{3m+2} + F_{3m-1} + \cdots + F_2) > 1$.

In Case 2 we replace $k = 3m$ with $k = 3m+1$ and in Case 3 we replace $k = 3m$ with $k = 3m+2$. The arguments are then the same as that of Case 1. □

Corollary 6. *Let $G(n) = (a_r, a_{r-1}, ..., a_2; a_1)$. If every $a_j = 3$ for some $r > j > 1$ but there exists at least one $a_q = 2$ such that $j > q \geq 1$, then for $p = T_j(n)$, $(n-p, 2p)$ is a winning position.*

Proof. Let $G(n) = (a_r, a_{r-1}, ..., a_2; a_1)$. Suppose every $a_j = 3$ for some $r > j > 1$ except for some $a_q = 2$ such that $j > q \geq 1$ and set $p = T_j(n)$. Define $G(n') = (b_r, b_{r-1}, ..., r_2; r_1)$ where each $b_j = 3$ for $r \geq j > 1$ and $b_1 = a_1$. Then, by definitions (2) and (4), if $T_q(n) = F_{i_q} + F_{i_{q-1}} + \cdots + F_{i_1}$ then $T_q(n) = F_{i_q - 1} + F_{i_{q-1} - 1} + \cdots + F_{i_1 - 1}$. If $i_1 - 1 = 1$, then $T_q(n')$ terminates with $F_{i_2 - 1}$, which will make no difference in the following argument. By lemma (7), $F_{i_{q+1}} - 2T_q(n') = g$ where $g \in \{1, 2\}$. By lemma (8), $F_{i_q + 1} \geq T_q(n') + 2$. Therefore, $F_{i_{q+1}} - 2T_q(n) \leq F_{i_{q+1}} - 2(T_q(n') + 2) = g - 4$. Since $g \in \{1, 2\}$, $g - 4 < 0$. This immediately shows that $T(n-p) = F_{i_{q+1}} \leq 2T_q(n) = 2p$ and hence $(n-p, 2p)$ is a winning position. □

We have now fully characterized when $T_j(n)$ is a winning move based soley on the $gap-vectors$ of n. We present a table below to summarize this section's findings. Let $n = F_{i_r} + F_{i_{r-1}} + \cdots F_{i_1}$. Then, $G(n) = (a_r, a_{r-1}, ..., a_2; a_1)$. Recall, each $a_j \geq 2$ by construction. Let the tail in question be $T_j(n)$. Then the "gaps" that surround F_{i_j} are precisely a_{j+1} and a_j. We have the following:

$\mathbf{a_{j+1}}$	$\mathbf{a_j}$	**Further Conditions**	**Winning Move**
≥ 3	≥ 2	$None$	Yes
2	≥ 4	$None$	Yes
2	2	$None$	No
2	3	$j \geq q \geq 1, a_q \geq 3$	Yes
2	3	$\exists q$ for $j \geq q \geq 1, a_q = 2$	No

Thus, by knowing the Zeckendorff representation of n, we may now find all possible winning moves, or moves that make $(n-p, 2p)$ a losing position.

4 Skilled vs Unskilled Players and Probabilities of an Unskilled Win

We begin this section by noting that in order for an unskilled player to win against a skilled player, (1) the unskilled player must go first and always make a winning move, or, (2) the skilled player must start from $n = F_k$ for some $n, k \in \mathbb{N}$. If not, the skilled player will immediately gain control of the game and provided the skilled player doesn't make any mistakes, he will force a win over the nonskilled player. It is from this perspective that we discuss probabilities of an unskilled win. For the remainder of this section, we assume that the unskilled player removes tokens randomly and that the skilled player is free from making errors. Further, we commit to the following strategy for a skilled player in a losing position:

Losing Position Strategy: (LPS) *if the skilled player is currently playing from a losing position, then he removes one token.*

Therefore, by definition (3) if the skilled player is given a position $(n, 2p')$ such that $T(n) > 2p'$, then set $p = 1$ and give the opponent the position $(n-1, 2)$. Hence, the unskilled player may take either one or two tokens on their next turn.

Lemma 9. *Let the current position be $(n, 2)$ to the unskilled player. Then for $p \in \{1, 2\}$, we have $P[(n-p, 2p) = \text{losing position}] \leq \frac{1}{2}$.*

Proof. Assume the unskilled player has position $(n, 2)$ where $n = F_{i_r} + F_{i_{r-1}} + \cdots + F_{i_1}$. If $F_{i_1} = 1 = F_2$, then $p = 1$ leaves $n - 1 = F_{i_r} + F_{i_{r-1}} + \cdots + F_{i_2}$ but $T(n-1) \geq F_4 = 3 > 2(1) = 2p$. Now, $p = 2$ leaves $(n-2, 4)$. Since $2 = p \neq T_j(n)$, by theorem (2), $(n-2, 4)$ is a winning position. Now suppose $F_{i_1} = 2 = F_3$, then the role of $p = 1$ and $p = 2$ are the reverse of case 1. Finally, If $F_{i_1} = m \geq 3$, then $T(n) = F_{i_1} > 2$ by ZRT. Then, by lemma (2), $(n-p, 2p)$ where $p = 1$ or $p = 2$ is a winning position. Hence, in all three instances, $P[(n-p, 2p) = \text{losing position}] \leq \frac{1}{2}$. □

Lemma 10. *Let $n = F_{i_r} + F_{i_{r-1}} + \cdots + F_{i_1}$, then $|G(n)| \leq \lfloor \frac{i_r}{2} \rfloor$.*

Proof. Let $n = F_{i_r} + F_{i_{r-1}} + \cdots + F_{i_1}$ and suppose $F_{i_r} = F_k$ from some k. Define $n' = F_{i_{r'}} + F_{i_{r-1'}} + \cdots + F_{i_{1'}}$ such that $F_{i_{r'}} = F_k$ and $G(n') = (2, 2, ..., 2; 2)$. Let $k = 2m$ for some $m \in \mathbb{N}$. Recall, every $a_j \geq 2$ by ZRT. Since there are $\frac{2m-2}{2} + 1 = m$ multiples of $2 \in [2, k]$, we have $m = \frac{k}{2} = |G(n')|$. Suppose $r > m$. Then by corollary (1) and definition (4), $r = |G(n)| > m$ implies that $F_{i_r} > F_k$ which is a contradiction. Now let $k = 2m + 1$. Note that $\lfloor \frac{k}{2} \rfloor = m$. Let n' be defined such that $G(n') = (a_{r'}, a_{r-1'}, ..., a_{2'}; a_{1'})$ where $F_{i_{r'}} = F_k$ and each $a_{j'} = 2$ except for some $a_{k'} = 3$ where $r' \geq k' \geq 1'$. Since there are $\lfloor \frac{(2m+1)-2}{2} + 1 \rfloor = m$ multiples of $2 \in [2, k]$, we have $m = \lfloor \frac{k}{2} \rfloor = |G(n')|$. Suppose $r > m$. Then by corollary (1) and definition (4), $r = |G(n)| > m$ implies that $F_{i_r} > F_k$ which is a contradiction. $\square$

Lemma (10) gives an upperbound on the number of terms in the Zeckendorff representation of some n.

Lemma 11. *For $k \geq 5$, $F_k \geq \frac{p^k - 0.1}{\sqrt{5}}$, where $p = \frac{\sqrt{5}+1}{2}$.*

Proof. The closed form of Fibonacci numbers is given by, $F_k = \frac{p^k - (-p)^{-k}}{\sqrt{5}}$, where $p = \frac{\sqrt{5}+1}{2}$, [1]. Then, we have $(-p)^{-5} \approx -0.09016994$ and $(-p)^{-6} \approx 0.05572809$. By simple application of the derivative test from elementary calculus, we see that this is a decreasing function for all $k \geq 5$. Hence, we have that $-0.1 \leq (-p)^{-k} \leq 0.1$ for all $k \geq 5$. Then for $k \geq 5$, we have $F_k \geq \frac{p^k - 0.1}{\sqrt{5}}$. $\square$

Corollary 7. *Let the current position be $(n, n-1)$ to the unskilled player where $n \geq 5$ and $F_{k+1} > n \geq F_k$, then $P[p = T_j(n)] \leq \frac{k\sqrt{5}}{2(p^k - 0.1)}$ where $1 \leq j \leq k$ and p is the unskilled players next move.*

Proof. If $n = F_k$, then $P[p = T_j(n)] = 0$ since the only tail is $F_k = n$ and the unskilled player may remove at most $n - 1$ tokens. Let $F_{k+1} > n > F_k$ so that the number of terms in the Zeckendorf representation of n is at most $\frac{k}{2}$ terms by lemma (10) and hence at most $\frac{k}{2}$ possible tails. Then, since there are at least F_k possible choices for p, by lemma (11) we have for $1 \leq j \leq k$, $P[p = T_j(n)] = \frac{k/2}{\frac{p^k - 0.1}{\sqrt{5}}} = \frac{k\sqrt{5}}{2(p^k - 0.1)}$. $\square$

Corollary (7) shows that if an unskilled player begins the game where $n \geq 5$, then the probability that the unskilled player chooses p such that p is a winning move is less than $\frac{2}{5}$ and by elementary calculus, the probability function $P[p = T_j(n)]$ can be shown to be a decreasing funtion for $k \geq 5$ so that as n increases, the probability that an unskilled player will choose a winning move from the beginning position (or any other of the form $(n, n-1)$) decreases exponentially. Note, if $n = 3$ then the first player will lose and if $n = 4$, then only winning move the first player may take is $p = 1$, thus the first player has a probability of $\frac{1}{4} < \frac{2}{5}$ of correctly choosing a tail.

We now have everything in place to state the main result of this section. This upperbound is dependent on the first move of the unskilled player however, and therefore cannot be calculated explicitly before the game begins.

Theorem 3. *Let $n > p$ and $(n - p, m)$ be the first position to the skilled player where $m \in \{n - 1, 2p\}$. Set $n' = n - p$ then using the* LPS*,*

1. *if $n \neq F_k$ for some $k \geq 4$ then $P[\text{Unskilled player wins}] \leq \frac{1}{5(2^{b-1})}$ where $b = \lfloor \frac{n'}{3} \rfloor$.*
2. *if $n = F_k$ for some $k \geq 5$, then $P[\text{Unskilled player wins}] \leq \frac{1}{2^b}$ where $b = \lfloor \frac{n'}{3} \rfloor$.*

Proof. There are two nontrivial cases needed to prove the result.

case 1: $n \neq F_k$ for some $n, k \in \mathbb{N}$. If the skilled player starts, he wins everytime. Thus, skilled player receives the position $(n - p, 2p)$ where $n - 1 \geq p \geq 1$. By LPS, after the initial turn, the unskilled player will always receive $(k, 2)$ for some $k < n$ and by lemma (9), $P[(n-p', 2p') = \text{losing position}] \leq \frac{1}{2}$ where $p' \in \{1, 2\}$. Hence, at most, 3 tokens are removed after one round of play. Let $n' = n - p$, then there will be at least $\lfloor \frac{n'}{3} \rfloor$ rounds played from this point in the game. By corollary (7) and repeated use of lemma (9), we find that $P[\text{Unskilled player wins}] \leq (\frac{2}{5})(\frac{1}{2^{\lfloor n'/3 \rfloor}}) = \frac{1}{5(2^{b-1})}$ where $b = \lfloor \frac{n'}{3} \rfloor$.

case 2: $n = F_k$ for some $n, k \in \mathbb{N}$. By lemma (2), removing p tokens make $(n-p, 2p)$ a winning position. Hence, the unskilled player loses if he goes first. Now assume the skilled player begins and by LPS, takes $1 < T(n)$ token. By lemma (2), $(n-1, 2)$ is a winning position. Thus, this position is that of case 1, where the unskilled player doesn't have the free move (n,n-1). Hence, $P[\text{Unskilled player wins}] \leq \frac{1}{2^{\lfloor n'/3 \rfloor}} = \frac{1}{2^b}$ where $b = \lfloor \frac{n'}{3} \rfloor$.

□

5 Final Remarks

In this paper we have characterized all winning algorithms for the game Fibonacci Nim. We have shown that the known winning algorithm is just a particular case of the generalized wining algorithm. In addition, we have shown an upperbound on the probability that an unskilled player may beat a skilled player if our unskilled player guesses randomly and our skilled player plays according to our losing position strategy.

Future research may look into different losing position strategies as well as different types of unskilled players. For example, as a second losing position strategy, by taking more than one token from a losing position, we may find a tighter upperbound on the probability that the unskilled player wins. Additionally, we could introduce a semi-skilled player, one whose guesses are not random but are based on some rule.

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6 Appendix

We now present these results for $n \in [1, 90] \subset \mathbb{N}$. First, recall the first 11 Fibonacci numbers: $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$, $F_7 = 13$, $F_8 = 21$, $F_9 = 34$, $F_{10} = 55$, $F_{11} = 89$. The column 'Zeck.' gives the Zeckendorf representation in binary form, where the rightmost number is the coeficient of F_2, for example, $17 = F_7 + F_4 + F_2 = (100101)$. The 'Moves' lists the sum of each winning tail. Continuing with $n = 17$, we have $G(17) = (3, 2; 2)$ and by the table above, we see that taking $F_2 = 1$ and $F_4 + F_2 = 3 + 1 = 4$ are both winning moves.

Fibonacci Nim Winning moves for $n \in [1, 90] \subset \mathbb{N}$

n	Zeck.	G(n)	Moves	n	Zeck.	G(n)	Moves
1	1	(2)	*none*	51	10100101	(2, 3, 2; 2)	1; 4
2	10	(3)	*none*	52	10101000	(2, 2; 5)	5
3	100	(4)	*none*	53	10101001	(2, 2, 3; 2)	1; 6
4	101	(2; 2)	1	54	10101010	(2, 2, 2; 3)	2
5	1000	(5)	*none*	55	100000000	(10)	*none*
6	1001	(3; 2)	1	56	100000001	(8; 2)	1
7	1010	(2; 3)	2	57	100000010	(7; 3)	2
8	10000	(6)	*none*	58	100000100	(6; 4)	3
9	10001	(4; 2)	1	59	100000101	(6, 2; 2)	1; 4
10	10010	(3; 3)	2	60	100001000	(5; 5)	5
11	10100	(2; 4)	3	61	100001001	(5, 3; 2)	1; 6
12	10101	(2, 2; 2)	1	62	100001010	(5, 2; 3)	2; 7
13	100000	(7)	13	63	100010000	(4; 6)	8
14	100001	(5; 2)	1	64	100010001	(4, 4; 2)	1; 9
15	100010	(4; 3)	2	65	100010010	(4, 3; 3)	2; 10
16	100100	(3; 4)	3	66	100010100	(4, 2; 4)	3; 11
17	100101	(3, 2; 2)	1; 4	67	100010101	(4, 2, 2; 2)	1; 12
18	101000	(2; 5)	5	68	100100000	(3; 7)	13
19	101001	(2, 3; 2)	1; 6	69	100100001	(3, 5; 2)	1; 14
20	101010	(2, 2; 3)	2	70	100100010	(3, 4; 3)	2; 15
21	1000000	(8)	*none*	71	100100100	(3, 3; 4)	3; 16
22	1000001	(6; 2)	1	72	100100101	(3, 3, 2; 2)	1; 4; 17
23	1000010	(5; 3)	2	73	100101000	(3, 2; 5)	5; 18
24	1000100	(4; 4)	3	74	100101001	(3, 2, 3; 2)	1; 6; 19
25	1000101	(4, 2; 2)	1; 4	75	100101010	(3, 2, 2; 3)	2; 20
26	1001000	(3; 5)	5	76	101000000	(2; 8)	21
27	1001001	(3, 3; 2)	5; 6	77	101000001	(2, 6; 2)	1; 22
28	1001010	(3, 2; 3)	2; 7	78	101000010	(2, 5; 3)	2; 23
29	1010000	(2; 6)	8	79	101000100	(2, 4; 4)	3; 24
30	1010001	(2, 4; 2)	1; 9	80	101000101	(2, 4, 2; 2)	1; 4; 25
31	1010010	(2, 3; 3)	2; 10	81	101001000	(2, 3; 5)	5; 26
32	1010100	(2, 2; 4)	3	82	101001001	(2, 3, 3; 2)	1; 6; 27
33	1010101	(2, 2, 2; 2)	1	83	101001010	(2, 3, 2; 3)	2; 7
34	10000000	(9)	*none*	84	101010000	(2, 2; 6)	8
35	10000001	(7; 2)	1	85	101010001	(2, 2, 4; 2)	1; 9
36	10000010	(6; 3)	2	86	101010010	(2, 2, 3; 3)	2; 10
37	10000100	(5; 4)	3	87	101010100	(2, 2, 2; 4)	3
38	10000101	(5, 2; 2)	1; 4	88	101010101	(2, 2, 2, 2; 2)	1
39	10001000	(4; 5)	5	89	1000000000	(11)	*none*
40	10001001	(4, 3; 2)	1; 6	90	1000000001	(9; 2)	1
41	10001010	(4, 2; 3)	2; 7	91	1000000010	(8; 3)	2
42	10010000	(3; 6)	8	92	1000000100	(7; 4)	3
43	10010001	(3, 4; 2)	1; 9	93	1000000101	(7, 2; 2)	1; 4
44	10010010	(3, 3; 3)	2; 10	94	1000001000	(6; 5)	5
45	10010100	(3, 2; 4)	3; 11	95	1000001001	(6, 3; 2)	1; 6
46	10010101	(3, 2, 2; 2)	1; 12	96	1000001010	(6, 2; 3)	2; 7
47	10100000	(2; 7)	13	97	1000010000	(5; 6)	8
48	10100001	(2, 5; 2)	1; 14	98	1000010001	(5, 4; 2)	1; 9
49	10100010	(2, 4; 3)	2; 15	99	1000010010	(5, 3; 3)	2; 10
50	10100100	(2, 3; 4)	3; 16	100	1000010100	(5, 2; 4)	3; 11