Factorization of Perplex Integers

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Abstract. The perplex numbers are a parallel-universe alternative to the complex numbers, a different two-dimensional algebra over $\mathbb{R}$. Instead of $\mathbf{i} = \sqrt{-1}$, there is $\mathbf{h} = \sqrt{1}$. $\mathbf{h}$ is a new square root of 1, equal to neither 1 nor -1. We explore the number theory of the integers within this parallel universe.

1. INTRODUCTION. In an 1843 letter, William Rowan Hamilton shared his discovery of the quaternion number system, which extends $\mathbb{C}$. Quaternions are expressed as $a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, where $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$, and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ do not commute. The quaternions have gone on to great renown, and are used in many contexts such as three-dimensional mechanics.

Less well known is the work of Hamilton's contemporary James Cockle. In 1848 he published (in [**5**]) his discovery of the tessarine number system (also called bicomplex by some later authors). These are also expressed as $a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, where $\mathbf{i}^2 = \mathbf{j}^2 = -1$, $\mathbf{k}^2 = 1$, and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ commute. This was followed in 1849 (in [**6**]) with the split-quaternions or coquaternions. These too are expressed as $a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, where $\mathbf{i}^2 = -1$, $\mathbf{j}^2 = \mathbf{k}^2 = 1$, and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ do not commute.

Tessarine and split-quaternion numbers have not been forgotten, and continue to be studied (e.g. in [**2, 9, 10, 11, 14**]). They too have applications, such as in signal processing. Polynomials over the tessarines admit a fundamental theorem of algebra (see [**13**]).

Taking a subalgebra generated by 1 and any one of $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in the quaternions just gives us the complex numbers. The same holds for tessarines and split-quaternions, for the generators which square to -1. With the other generators, however, we get something new. Let us fix a generator $\mathbf{h}$, satisfying $\mathbf{h}^2 = 1$, and consider the numbers $a + b\mathbf{h}$. These are called perplex numbers[1].

Perplex numbers have become more popular lately. They have applications in physics ([**7, 8**]), algebra ([**12, 13**]), geometry ([**15**]), and dynamical systems ([**16**]). In this note, we propose to study the perplex integers, from an algebraic number theory perspective.

There are two natural definitions for "perplex integers" (neither of which appear to have been studied before). We will consider both. The first is $\mathbb{P}_1 = \{a + b\mathbf{h} : (a, b) \in \mathbb{Z}^2\}$; these were of interest in [**15**]. To define the second, we first define the checkerboard space $\mathbb{X} = \{(a, b) \in \mathbb{Z}^2 : a \equiv b \pmod 2\}$, and $\mathbf{H} = \frac{1+\mathbf{h}}{2}$. We mention in passing that $\mathbf{H}^2 = \left(\frac{1+\mathbf{h}}{2}\right)^2 = \frac{2+2\mathbf{h}}{4} = \mathbf{H}$. We can now define $\mathbb{P}_2 = \{\frac{a}{2} + \frac{b}{2}\mathbf{h} : (a, b) \in \mathbb{X}\} = \{a + b\mathbf{H} : (a, b) \in \mathbb{Z}^2\}$.

$\mathbb{P}_2$ is natural because it is the intersection of the ring $\mathbb{Q}(\mathbf{h})$ with the set of quadratic integers $\{x : x^2 + bx + c = 0,\ \text{with } b, c \in \mathbb{Z}\}$. $\mathbb{P}_1$ is an order in the ring of quadratic integers $\mathbb{P}_2$, specifically $\mathbb{P}_1 = \{2a + 2b\mathbf{H} : (a, b) \in \mathbb{Z}^2\}$.

[1] These have also been called (see [**1**]) split complex numbers, hyperbolic numbers, hyperbolic complex numbers, double numbers, real tessarines, algebraic motors, bireal nubmers, approximate numbers, counter-complex numbers, anormal-complex numbers, Lorentz numbers, paracomplex numbers, semi-complex numbers, split binarions, spacetime numbers, Study numbers, and twocomplex numbers. They have been rediscovered frequently!

It is very common to study rings of quadratic integers in algebraic number theory, with $\mathbf{h}$ replaced by $\sqrt{D}$ for some $D \in \mathbb{Z}$. Traditionally D is assumed to be squarefree, and that $\mathbb{Q}(\sqrt{D})$ is a subfield of $\mathbb{C}$. These assumptions cause the ring of quadratic integers to be a domain, i.e. have no zero divisors. One famous example is $D = -1$, which gives the Gaussian integers $\{a + b\mathbf{i} : a, b \in \mathbb{Z}\}$.

In our context, however, $\mathbb{Q}(\mathbf{h})$ is not a subfield of $\mathbb{C}$ (or a field at all), and $(1 + \mathbf{h})(1 - \mathbf{h}) = 0 = \mathbf{H}(1 - \mathbf{H})$. Hence, both $\mathbb{P}_1$ and $\mathbb{P}_2$ have zero divisors. To study factorizations here we need some special tools in this non-domain context; for a nice introduction to these tools see [**3**].

Let x, y be nonzero elements of a commutative ring with zero divisors. We say they are associates if each divides the other; we say they are strong associates if $x = yz$ for some unit z; we say they are very strong associates if they are associates yet $x \neq yz$ for any non-unit z. As the choice of names implies, being very strong associates implies being strong associates, which in turn implies being associates.

We say that x is irreducible (resp. strongly irreducible, very strongly irreducible) if $x = yz$ means that either y or z is an associate (resp. strong associate, very strong associate) of x. Hence the familiar notion of irreducibility has now been split into a hierarchy of three notions. Being very strongly irreducible implies being strongly irreducible, which in turn implies being irreducible. We say that x is prime as per usual, i.e. if $x|yz$ then $x|y$ or $x|z$. Being prime implies being irreducible, but no further. By "reducible" we mean possessing none of the three irreducibility properties.

2. SPLIT NORM. Our main tool for studying factorization of perplex integers will be the split norm function, defined as $S(a + b\mathbf{h}) = (a + b, a - b)$. Note that whether $a + b\mathbf{h} \in \mathbb{P}_1$ (and $(a, b) \in \mathbb{Z}^2$) or $a + b\mathbf{h} \in \mathbb{P}_2$ (and $(2a, 2b) \in \mathbb{X}$), $S(a + b\mathbf{h}) \in \mathbb{Z}^2$. This will allow S to be useful for both contexts. We will consider $\mathbb{Z}^2$ as a ring ($\mathbb{X}$ is a subring), with coordinatewise addition and multiplication, i.e. $(a, b) + (c, d) = (a + c, b + d)$ and $(a, b)(c, d) = (ac, bd)$. Basic properties of the split norm S follow.

Proposition 1. *Let $S : \mathbb{P}_1 \to \mathbb{Z}^2$ (resp. $S : \mathbb{P}_2 \to \mathbb{Z}^2$). Then:*

1. *For all $x, y \in \mathbb{P}_1$ (resp. $x, y \in \mathbb{P}_2$), $S(x + y) = S(x) + S(y)$.*
2. *For all $x, y \in \mathbb{P}_1$ (resp. $x, y \in \mathbb{P}_2$), $S(xy) = S(x)S(y)$.*
3. *S is injective.*
4. *$Im(S) = \mathbb{X}$ (resp. $Im(S) = \mathbb{Z}^2$), where $Im(S)$ denotes the image of S.*
5. *$S : \mathbb{P}_1 \to \mathbb{X}$ (resp. $S : \mathbb{P}_2 \to \mathbb{Z}^2$) is a ring homomorphism.*

Proof. (1) and (2) can be proved with direct calculation:
$S(x + y) = S(a + c + (b + d)\mathbf{h}) = (a + c + b + d, a + c - (b + d)) = (a + b, a - b) + (c + d, c - d) = S(a + b\mathbf{h}) + S(c + d\mathbf{h}) = S(x) + S(y)$, and
$S(xy) = S((a + b\mathbf{h})(c + d\mathbf{h})) = S(ac + bd + (ad + bc)\mathbf{h}) = (ac + bd + ad + bc, ac + bd - ad - bc) = (a + b, a - b)(c + d, c - d) = S(a + b\mathbf{h})S(c + d\mathbf{h}) = S(x)S(y)$.

Now, let $(m, n) \in \mathbb{Z}^2$. Then $S(a + b\mathbf{h}) = (m, n)$ gives the system of equations $\{a + b = m, a - b = n\}$ with unique solution (in $\mathbb{Q}$) of $\{a = \frac{m+n}{2}, b = \frac{m-n}{2}\}$. The uniqueness gives (c). Note that, for any m, n, we have $m + n \equiv m - n \pmod 2$. Hence, every preimage $\{\frac{m+n}{2} + \frac{m-n}{2}\mathbf{h}\}$ will be in $\mathbb{P}_2$. However, not every preimage will be in $\mathbb{P}_1$; for this, we need $(\frac{m+n}{2}, \frac{m-n}{2}) \in \mathbb{Z}^2$. This occurs exactly when $m \equiv n \pmod 2$, i.e. when $(m, n) \in \mathbb{X}$.

Finally, note that $S(1 + 0\mathbf{h}) = (1, 1)$. Also, $1 + 0\mathbf{h}$ is the multiplicative identity in $\mathbb{P}_1$ and $\mathbb{P}_2$, while $(1, 1)$ is the multiplicative identity in $\mathbb{X}$ and $\mathbb{Z}^2$. ■

We mention in passing the norm function $N(a + b\mathbf{h}) = a^2 - b^2$, from $\mathbb{P}_1$ (resp. $\mathbb{P}_2$) to $\mathbb{Z}$, which is also totally multiplicative like S. This is the usual tool used to study rings of quadratic integers, but for us it is more useful to exploit the identity $a^2 - b^2 = (a + b)(a - b)$, which motivates the split norm.

3. FACTORIZATION. Applying the ring homomorphism provided by Proposition 1, we can study factorization properties in the simpler rings $\mathbb{X}$ and $\mathbb{Z}^2$ — these same properties must hold in $\mathbb{P}_1$ and $\mathbb{P}_2$. We will now develop these properties in Propositions 2 and 4. Our main results, Theorems 3 and 5, follow as corollaries.

Let $\mathcal{P}$ denote the set of rational primes $\{2, 3, 5, 7, 11, \ldots\}$, and $\mathcal{P}^\star = \mathcal{P} \setminus \{2\}$. Multiple instances of $\pm$ should be considered independent. For example, $(\pm 1, \pm 1)$ represents four elements of $\mathbb{Z}^2$ (also $\mathbb{X}$). For any $(a, b) \in \mathbb{Z}^2$ (or $\mathbb{X}$), we say that its friends are $\{(\pm a, \pm b)\} \cup \{(\pm b, \pm a)\}$. Its friends are found by multiplying by a unit and/or reversing coordinates. Note that (a, b) is irreducible (strongly irreducible, very strongly irreducible, prime, reducible) if and only if its friends are too. We now present factorization properties of $\mathbb{Z}^2$.

Proposition 2. *Consider the ring* $\mathbb{Z}^2$.

1. *There are four units:* $(\pm 1, \pm 1)$*, i.e.* $(1, 1)$ *and its friends.*
2. *The zero divisors are* $\{(a, 0) : a \in \mathbb{Z} \setminus \{0\}\} \cup \{(0, b) : b \in \mathbb{Z} \setminus \{0\}\}$.
3. *Every associate is a strong associate; every irreducible is a strong irreducible.*
4. *The very strong irreducibles are* $\{(1, p) : p \in \mathcal{P}\}$ *and their friends.*
   The other (strong but not very strong) irreducibles are $(1, 0)$ *and its friends.*
5. *The irreducibles are all prime.*

Proof. (1) If (a, b) is a unit, then there is some (c, d) with $(a, b)(c, d) = (1, 1)$. Hence $ac = 1$ and $bd = 1$, as integers. So a, c are units in $\mathbb{Z}$, which are $+1, -1$.
(2) If (a, b) is a zero divisor, then it is nonzero and there is some nonzero (c, d) with $(a, b)(c, d) = (0, 0)$. Hence $ac = 0$ and $bd = 0$, as integers. Since (c, d) is nonzero, then $c \neq 0$ (hence $a = 0$) or $d \neq 0$ (hence $b = 0$).
(3) If $(a, b)|(c, d)$ and $(c, d)|(a, b)$, then: $a|c$, $c|a$, $b|d$, $d|b$ as integers. Hence, $a = \pm c$ and $b = \pm d$, so $(a, b) = (\pm 1, \pm 1)(c, d)$.
(4) Now consider $(a, b) \in \mathbb{Z}^2$ with $ab \neq 0$. If neither a nor b is ± 1, then $(a, b) = (a, 1)(1, b)$ shows that (a, b) is reducible. If $a = \pm 1$ and b is reducible in $\mathbb{Z}$, then $b = \pm b_1 b_2$ for two non-units b_1, b_2. Now $(a, b) = (a, b_1), (1, b_2)$ shows that (a, b) is reducible. The case $(a_1 a_2, \pm 1)$ is similarly reducible.

Next, consider $x = (1, p)$, for some $p \in \mathcal{P}$. If $(a, b)|x$, then $a|1$ and $b|p$. Then $a = \pm 1$, and either $b = \pm 1$ (in which case (a, b) is a unit) or $b = \pm p$ (in which case (a, b) is an associate of x). This proves that x is irreducible.

Note that $(a, 0) = (a, 1)(1, 0)$, so if $a \neq \pm 1$ then $(a, 0)$ is reducible. If instead $a = 1$ and $(a, 0) = (b, c)(m, n)$, then $bm = 1$ and $cn = 0$. Hence (b, c) or (m, n) is $(\pm 1, 0)$, which is an associate of $(a, 0)$. Hence $(1, 0)$ is irreducible.

Lastly, we consider strong vs. very strong. Suppose that $(1, p)$ and (c, d) are associates, and $(1, p) = (c, d)(m, n)$. Since $(1, p)$ is a strong irreducible, $(c, d) = (\pm 1, \pm p)$. But now $(m, n) = (\pm 1, \pm 1)$, a unit. Hence $(1, p)$ is a very strong irreducible. However, $(1, 0)$ and $(1, 0)$ are associates, yet $(1, 0) = (1, 0)(1, 3)$ for the non-unit $(1, 3)$, so $(1, 0)$ is not a very strong irreducible.
(5) Suppose first that $(1, p)(a, b) = (c, d)(m, n)$ for some $p \in \mathcal{P}$. Then $pb = dn$. Since p is prime, without loss we assume $p|d$. Now, $(c, d) = (1, p)(c, \frac{d}{p})$. Hence $(1, p)$ is prime. Next, suppose that $(1, 0)(a, b) = (c, d)(m, n)$. Now $dn = 0$, so without loss we assume $d = 0$. Now, $(c, d) = (1, 0)(c, d)$. Hence $(1, 0)$ is prime. ∎

Proposition 2 tells us that $\mathbb{Z}^2$ is a ring where every irreducible is prime. Every nonzero divisor has a factorization into irreducibles, unique up to order and associates. However, zero divisors in $\mathbb{Z}^2$ lack this property: $(2,1)(1,0) = (2,1)(1,0)(1,7)$.

We mention as an aside yet a fourth definition of irreducibility. We say that x is m-irreducible if $x = yz$ means that either y is either a unit or associated to x. This turns out to be intermediate in strength between very strong irreducibles and strong irreducibles. In $\mathbb{Z}^2$, we show the strong irreducible $(1,0)$ and its three friends are not m-irreducible by noting $(1,0) = (1,3)(1,0)$, where $(1,3)$ is neither a unit nor associated to $(1,0)$.

We now use S to pull the results of Proposition 2 back to $\mathbb{P}_2$. Now, the friends of $a + b\mathbf{h}$ are $\{\pm a \pm b\mathbf{h}\} \cup \{\pm b \pm a\mathbf{h}\}$.

Theorem 3. *Consider the perplex integers $\mathbb{P}_2$. Then:*

1. *There are four units: $\pm 1, \pm \mathbf{h}$.*
2. *The zero divisors are $\{\frac{a}{2} + \frac{a}{2}\mathbf{h} : (a,a) \in \mathbb{X} \setminus \{(0,0)\}\}$ and their friends.*
3. *Every associate is a strong associate; every irreducible is a strong irreducible.*
4. *The very strong irreducibles are: $\{\frac{p+1}{2} + \frac{p-1}{2}\mathbf{h} : p \in \mathcal{P}\}$ and their friends.*
   The other (strong but not very strong) irreducibles are: $\pm\frac{1}{2} \pm \frac{1}{2}h$.
5. *The irreducibles are all prime.*

Note that all very strong irreducibles, with the sole exception of those corresponding to $p = 2$, are also in $\mathbb{P}_1$, since any odd prime p leads to $\frac{p+1}{2}, \frac{p-1}{2} \in \mathbb{Z}$.

It's interesting to compare $\mathbb{P}_2$ with the well-known Gaussian integers $\mathbb{Z}[i] = \{a + b\mathbf{i} : (a,b) \in \mathbb{Z}^2\}$. These too satisfy the property that every irreducible is prime. Since $\mathbb{Z}[i]$ is a domain, this gives unique factorization. It has a norm $N(a + b\mathbf{i}) = a^2 + b^2$. Its irreducibles are of three types. Those $p \in \mathcal{P}$ congruent to 1 modulo 4 give rise to irreducible $a + b\mathbf{i}$ with $a^2 + b^2 = p$ and each of a, b nonzero. Those $p \in \mathcal{P}$ congruent to 3 modulo 4 give rise to irreducible $p + 0\mathbf{i}$ (as well, of course, this multiplied by any of the units $\pm 1, \pm i$). Lastly, each of $\pm 1 \pm \mathbf{i}$ are irreducible. To summarize, in $\mathbb{Z}[i]$, $2 = (1+i)(1-i)$ is ramified, primes congruent to 3 are inert, and primes congruent to 1 are decomposed. In contrast, every rational prime is decomposed in $\mathbb{P}_2$.

We now turn to $\mathbb{P}_1$, which has a simpler definition but more complicated factorization properties, as compared with $\mathbb{P}_2$. Recall that S gives a ring isomorphism between $\mathbb{P}_1$ and $\mathbb{X}$. By $\mathbb{N}$ we mean the positive integers, and by $\mathbb{N}_0$ the nonnegative integers.

Proposition 4. *Consider the ring $\mathbb{X}$.*

1. *There are four units: $(\pm 1, \pm 1)$.*
2. *The zero divisors are $\{(2a, 0) : a \in \mathbb{Z} \setminus \{0\}\} \cup \{(0, 2b) : b \in \mathbb{Z} \setminus \{0\}\}$.*
3. *Every associate is a strong associate; every irreducible is a strong irreducible.*
4. *The very strong irreducibles are of two types, odd and even:*
   the odd ones are $\{(1,p) : p \in \mathcal{P}^\star\}$ and their friends;
   the even ones are $\{(2, 2^n) : n \in \mathbb{N}\}$ and their friends.
   The other (strong but not very strong) irreducibles are $(2,0)$ and its friends.
5. *All irreducibles are prime, except the even very strong ones (which are not).*

Proof. Much of the proof of Proposition 2 carries over: all of (1)-(3); also, that every irreducible (a,b) with $ab \neq 0$ is very strong; also, that $(1,p)$ with $p \in \mathcal{P}^\star$ (and its friends) is irreducible and prime; also, that (a,b) with a,b odd is reducible unless (a,b) is $(1,p)$ for some $p \in \mathcal{P}^\star$ (or one of its friends).

Now consider (a, b) with a, b even and nonzero. If $\pm a$ is not a power of 2, then write $a = \pm 2^m t$ (t odd) and $(a, b) = (\frac{a}{t}, b)(t, 1)$ shows that (a, b) is reducible. The case of $\pm b$ not a power of 2 is similar. If instead $a = \pm 2^m, b = \pm 2^n$ with $m, n \geq 2$, then $(a, b) = (2, 2)(\frac{a}{2}, \frac{b}{2})$ and again (a, b) is reducible.

Now, consider $x = (2, 2^n)$. If $(a, b)(c, d) = x$, then $ac = 2$ and $bd = 2^n$. Since $(a, b) \in \mathbb{X}$, $a \equiv b \pmod 2$. Hence if $a = \pm 1$, then $b = \pm 1$ and (a, b) is a unit. If instead $a = \pm 2$, then $c = \pm 1$. Since $(c, d) \in \mathbb{X}$, we must have $c \equiv d \pmod 2$, so $d = \pm 1$ and then (c, d) is a unit. This proves that x is irreducible.

If $(a, 0) \in \mathbb{X}$, then a is even. We then write $(a, 0) = (2, 0)(\frac{a}{2}, b)$ (where b is chosen to be 1 or 2 based on parity of $\frac{a}{2}$), which shows that $(a, 0)$ is reducible unless $a = \pm 2$.

Suppose now that $(2, 0)(a, b) = (c, d)(m, n)$, then $cm = 2a$ and $dn = 0$. Without loss, suppose that $d = 0$. Now, since $(c, d) \in \mathbb{X}$, we must have c even. Hence $(c, d) = (2, 0)(\frac{c}{2}, t)$ (where t is chosen to be 1 or 2 based on parity of $\frac{c}{2}$). This proves that $(2, 0)$ is prime, and hence also irreducible.

Let $x = (2, 2^n)$, and take $y = (2, 2^n)(2, 2^{3n}) = (2, 2^{n+1})(2, 2^{3n-1})$. x divides y, but divides neither $(2, 2^{n+1})$ nor $(2, 2^{3n-1})$ (because $n \neq n + 1$ and $n \neq 3n - 1$). This proves that neither x, nor its friends, is prime. ■

Proposition 4 tells us that $\mathbb{X}$ (and hence $\mathbb{P}_1$) does *not* have unique factorization, even among the non-zero-divisors. In fact, it is very far from this. For any $n \in \mathbb{N}$ with $n \geq 2$, we set $x = (2, 2)^n = (2^n, 2^n) = (2^{n-1}, 2)(2, 2^{n-1})$. Hence x has a very long factorization into n irreducibles, and also a factorization into 2 irreducibles. The elasticity of x is therefore the ratio $\frac{n}{2}$, and the elasticity of $\mathbb{X}$ (and thus $\mathbb{P}_1$) is the supremum of these, i.e. infinite. For a lively introduction to factorization theory, including terms such as "elasticity", see **[4]**.

We can again use S to pull the results back to $\mathbb{P}_1$. We retain the terms "odd" and "even" from Proposition 4, although the parity is somewhat obscured.

Theorem 5. *Consider the perplex integers $\mathbb{P}_1$. Then:*

1. *There are four units: $\pm 1, \pm \mathbf{h}$.*
2. *The zero divisors are $\{a + a\mathbf{h} : (a, a) \in \mathbb{Z}^2 \setminus \{(0, 0)\}\}$ and their friends.*
3. *Every associate is a strong associate; every irreducible is a strong irreducible.*
4. *The very strong irreducibles are of two types, odd and even:*
   the "odd" ones are $\{\frac{p+1}{2} + \frac{p-1}{2}\mathbf{h} : p \in \mathcal{P}^\star\}$ and their friends;
   the "even" ones are $\{2^n + 1 + (2^n - 1)\mathbf{h} : n \in \mathbb{N}_0\}$ and their friends.
   The other (strong but not very strong) irreducibles are $(2, 0)$ and its friends.
5. *All irreducibles are prime, except the even very strong ones (which are not).*

Note that every irreducible $(a, b) \in \mathbb{P}_1$ satisfies $|a| + |b| \in \mathcal{P}^\star \cup \{2^n : n \in \mathbb{N}\}$ and $1 \leq ||a| - |b|| \leq 2$; these properties can be seen to characterize the irreducibles.

4. FURTHER WORK. These results are only a first step toward understanding factorization in a perplex world. Recall that the Gaussian integers are the simplest quadratic integer ring $\mathbb{Z}[\sqrt{-1}]$. Traditionally, there are infinitely more of great interest: $\mathbb{Z}[\sqrt{D}]$ (for D squarefree and congruent to 1 modulo 4) and $\mathbb{X}[\frac{\sqrt{D}}{2}]$ (for D squarefree and congruent to 3 modulo 4). Similarly, the perplex integers are the simplest example of a quadratic integer ring with square D. One could ask similar questions about $\{a + 2b\mathbf{h} : a, b \in \mathbb{Z}\}$, corresponding to $D = 4$. Further, one could look at integer subrings of $\mathbb{R} + \mathbb{R}\mathbf{h}$, such as the algebraic integers contained in $\mathbb{Q}(\sqrt{2}\mathbf{h})$.

Also, Gaussian integers are isomorphic to the quotient ring $\mathbb{Z}[x]/(x^2 + 1)$. Similarly, $\mathbb{P}_1$ is isomorphic to the quotient ring $\mathbb{Z}[x]/(x^2 - 1)$, and $\mathbb{P}_2$ to $\mathbb{Z}[x]/(x^2 - x)$.

One could similarly study factorization in quotient rings $\mathbb{Z}[x]/(p(x))$ for other reducible polynomials $p(x)$.

ACKNOWLEDGMENT. The author wishes to gratefully acknowledge the anonymous referees and the editors, whose efforts greatly improved this work.

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