SQUAREFREE DIVISOR COMPLEXES OF CERTAIN NUMERICAL SEMIGROUP ELEMENTS

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ABSTRACT. A numerical semigroup S is an additive subsemigroup of the non-negative integers with finite complement, and the squarefree divisor complex of an element $m \in S$ is a simplicial complex Δ_m that arises in the study of multigraded Betti numbers. We compute squarefree divisor complexes for certain classes numerical semigroups, and exhibit a new family of simplicial complexes that occur as the squarefree divisor complex of some numerical semigroup element.

1. Introduction

A numerical semigroup is a cofinite subset $S \subseteq \mathbb{Z}_{\geq 0}$ that is closed under addition. The numerical semigroup generated by $\{t_1, t_2, \ldots, t_k\} \subset \mathbb{Z}_{\geq 0}$ is the set

$$\langle t_1, t_2, \ldots, t_k \rangle = \{\alpha_1 t_1 + \cdots + \alpha_k t_k : \alpha_i \in \mathbb{Z}_{\geq 0}\}.$$

If $t \in S$ is not a sum of strictly smaller elements of S, we call t irreducible in S. The set $\{n_1, n_2, \ldots, n_d\}$ of irreducible elements of S is the unique minimal generating set of S, i.e. $S = \langle n_1, n_2, \ldots, n_d \rangle$, and no proper subset generates S. Unless otherwise stated, for the remainder of the document, whenever we write $S = \langle n_1, \ldots, n_d \rangle$, we assume $n_1, \ldots, n_d$ are precisely the irreducible elements of S. In this case, we call d the embedding dimension of S.

Given numerical semigroup S, the Frobenius number of S, denoted $\mathrm{F}(S)$, is the largest element of $\mathbb{Z} \setminus S$. We call $x \in \mathbb{Z} \setminus S$ a pseudo-Frobenius number if $x + s \in S$ for all $s \in S \setminus \{0\}$, and denote by $\mathrm{PF}(S)$ the set of all such x, noting in particular that $\mathrm{F}(S) \in \mathrm{PF}(S)$. For a more thorough introduction to numerical semigroups, see [19].

Fix a numerical semigroup $S = \langle n_1, n_2, \ldots, n_d \rangle$, and let $[d] = \{1, \ldots, d\}$. A simplicial complex is a collection $\Delta \subset 2^{[d]}$ of subsets of $[d]$ (called faces or simplices) such that if $F \in \Delta$ and $F' \subset F$, then $F' \in \Delta$. The maximal faces of Δ are called facets. We define the Euler characteristic of Δ as

$$\chi(\Delta) = \sum_{F \in \Delta} (-1)^{|F|}.$$

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For each face $F \subset [d]$, write $n_F = \sum_{i \in F} n_i$. Given $m \in S$, we define the squarefree divisor complex of m in S to be the simplicial complex

$$\Delta_m^S = \{F \in [d] : m - n_F \in S\}.$$

When there can be no confusion, we omit the superscript and write Δ_m instead of Δ_m^S. For a full treatment on simplicial complexes, see [20, Section 2].

Squarefree divisor complexes were defined in [7] for studying multigraded Betti numbers of semigroup rings (although the complexes themselves appeared previously in [21], wherein they were called shadow sets). Specifically, they prove the Hilbert series of a numerical semigroup S (see [2]) can be expressed as

$$\mathcal{H}(S;t) = \sum_{m \in S} t^m = \frac{\sum_{m \in S} \chi(\Delta_m) t^m}{(1 - t^{n_1}) \cdots (1 - t^{n_d})}.$$

Note that each squarefree divisor complex Δ_m is homotopy equivalent to simplicial complex ∇_m, due to Eliahou [16], that has a larger vertex set (one for each expression of m as a sum of generators). In particular, the number of vertices of ∇_m grows unbounded as m increases within S, while the number of vertices of Δ_m remains constant.

Hilbert series have long been used to study numerical semigroups [4]. There has also been a recent surge of interest [1, 5, 12, 15] yielding, among other things, characterizations of the numerator above in several cases [9, 11, 17]. The authors of [7] also pose the question "which simplicial complexes Δ are realizable as the squarefree divisor complex Δ_m^S for some S and some $m \in S$", and provided a complete answer in the case where Δ is a graph.

Our contribution is twofold. First, we provide a novel method of iteratively constructing squarefree divisor complexes and use it to produce a new family of realizable squarefree divisor complexes. Second, for several classes of numerical semigroups, we explicitly compute the squarefree divisor complex of every element m for which Δ_m has nonzero Euler characteristic (i.e. those appearing in the numerator of the Hilbert series equation above). For each class of semigroups, the elements with nonzero Euler characteristic are known [9], but the original proof used high level machinery rather than squarefree divisor complexes. Our work provides an alternative proof "from first principles" in these cases.

2. Realization of simplicial complexes

The main result of this section is Corollary 2.8, which states that any fat forest (Definition 2.1) can be realized as the squarefree divisor complex of some numerical semigroup element. Our proof comes in two steps. First, we give a method to produce disjoint unions of squarefree divisor complexes (Theorem 2.4) using a construction called gluing (defined in [10], though some authors had used it earlier). Our gluing construction also appeared as [7, Lemma 2.6(a)], though it was presented without

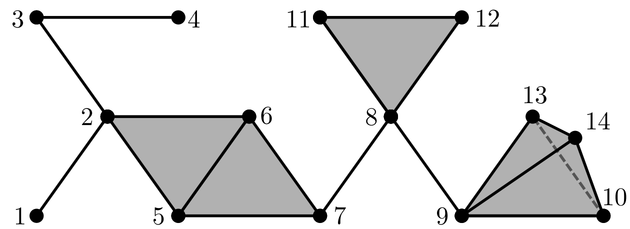

FIGURE 1. An example of a fat tree.

proof and with insufficiently strong hypotheses (see Example 2.5). Second, we iterate a construction called "inflation" to build any fat tree one vertex at a time.

Definition 2.1. A simplicial complex Δ is a *fat tree* if its facets $F_1, F_2, \ldots$ can be ordered in such a way that the intersection of each F_j with $F_1 \cup \cdots \cup F_{j-1}$ is a simplex. We say Δ is a *fat forest* if it is a disjoint union of fat trees.

Example 2.2. Intuitively, any fat tree Δ can be constructed one facet at a time by attaching each new facet along an existing face. In doing so, no cycles can be formed. Figure 1 depicts one such complex, whose vertices are labeled in accordance with one possible facet order.

Lemma 2.3 ([19, Lemma 8.8])**.** *Fix numerical semigroups $S = \langle n_1, n_2, \ldots, n_d \rangle$, and $S' = \langle n_1', n_2', \ldots, n_{d'}' \rangle$. If $k \in S \setminus \{0, n_1, \ldots, n_d\}$ and $k' \in S' \setminus \{0, n_1', \ldots, n_{d'}'\}$ satisfy $\gcd(k, k') = 1$, then the numerical semigroup*

$$k'S + kS' = \langle k'n_1, k'n_2, \ldots, k'n_d, kn_1', kn_2', \ldots, kn_{d'}' \rangle$$

has embedding dimension $d + d'$.

Theorem 2.4 (Disjoint union)**.** *Fix numerical semigroups $S = \langle n_1, n_2, \ldots, n_d \rangle$ and $S' = \langle n_1', n_2', \ldots, n_{d'}' \rangle$. Given coprime $k \in S$ and $k' \in S'$ that are not irreducible, and letting $T = k'S + kS'$, we have*

$$\Delta_{kk'}^T = \Delta_k^S \cup (\Delta_{k'}^{S'} + d),$$

a disjoint union.

Proof. First, note that the union is disjoint as $\Delta_k^S \subseteq 2^{[d]}$ while $\Delta_{k'}^{S'} + d \subseteq 2^{[d+d'] \setminus [d]}$. If $F \in \Delta_k^S$, then $k'k - k'n_F = k'(k - n_F) \in S$, so $kk' - n_F \in T$, meaning $F \in \Delta_{kk'}^T$. By symmetry, this implies $\Delta_{kk'}^T \supseteq \Delta_k^S \cup (\Delta_{k'}^{S'} + d)$.

Conversely, fix $F \in \Delta_{kk'}^T$. Suppose F contains elements from both $[d]$ and $[d+d'] \setminus [d]$. We can conclude $\{a, b\} \in \Delta_{kk'}^T$ for some $a \in [d]$ and $b \in [d + d'] \setminus [d]$. This means $kk' - k'n_a - kn_{b-d}' \in T$, so we must have

$$kk' = k'(n_a + m) + k(n_{b-d}' + m')$$

for some $m \in S$ and $m' \in S'$. Taking the above equality modulo k, we see that $k \mid (n_a + m)$, and since $n_a \neq 0$, we must have $n_a + m \geq k$. But now

$$k'(n_a + m) + k(n'_{b-d} + m') \geq k'k + kn'_{b-d} > k'k,$$

which is a contradiction.

Now, this means any $F \in \Delta^T_{kk'}$ must be entirely contained in $[d]$ or in $[d + d'] \setminus [d]$. By symmetry, it suffices to address the case $F \subseteq [d]$, wherein $k'k - k'n_F \in T$. For some $m \in S$ and $m' \in S'$, we have $kk' - k'n_F = k'm + km'$. Taking both sides modulo k', we conclude $k' \mid m'$. If $m' > 0$, then $m' \geq k'$, which yields a contradiction since then we would have $kk' \geq k'n_F + km' \geq k'n_F + kk'$. Hence $m' = 0$ and dividing by k' yields $k - n_F = m \in S$, so $F \in \Delta^S_k$. By symmetry, we conclude $\Delta^T_{kk'} = \Delta^S_k \cup (\Delta^{S'}_{k'} + d)$. □

Example 2.5. The "coprime" hypothesis in Theorem 2.4 can't be omitted. For instance, if $k = 2 \in S = \langle 2, 5\rangle$ and $k' = 6 \in S' = \langle 6, 10, 15\rangle$, then we have $T = k'S + kS' = \langle 12, 20, 30\rangle$ and Δ^S_2, $\Delta^{S'}_6$ and Δ^T_{12} each have exactly one vertex.

Theorem 2.6. *Fix a numerical semigroup $S = \langle n_1, \ldots, n_d\rangle$, an element $m \in S$, and a set $F \subset [d]$. Let $b = m - n_F$, fix $p \in \mathbb{Z}_{\geq 0}$ with $\gcd(p, b) = 1$, and write $T = pS + b\langle 1\rangle$. If $b \in S \setminus \{0, n_1, \ldots, n_d\}$ and $\Delta^S_{n_F} = 2^F$, then the squarefree divisor complex Δ^T_{pm} contains all faces of the complex Δ^S_m with one new vertex b and one new facet $F \cup \{b\}$.*

Proof. By Lemma 2.3, each pn_i is a minimal generator of T, so $\Delta^S_m \subset \Delta^T_{pm}$, and since

$$pm = p(m - n_F) + pn_F = pb + \sum_{i \in F} pn_i,$$

we have $F \cup \{b\} \in \Delta^T_{pm}$. Now, we claim any face of Δ^T_{pm} containing b lies in $F \cup \{b\}$. Indeed, any factorization of pm involving b can be written as

$$pm = a_0 b + \sum_{i=1}^{d} a_i p n_i = a_0(m - n_F) + \sum_{i=1}^{d} a_i p n_i,$$

and taking both sides modulo p implies $p \mid a_0$. Writing $a_0 = kp$ for $k \geq 1$, dividing by p, and adding $n_F - m$ to both sides yields, we have

$$n_F = (k - 1)(m - n_F) + \sum_{i=1}^{d} a_i n_i.$$

Since $b = m - n_F \in S$, the above yields a factorization for n_F, but since $\Delta_{n_F} = 2^F$, this means $a_i = 0$ for each $i \notin F$. This completes the proof. □

Theorem 2.7. *Any fat tree occurs as the squarefree divisor complex of some numerical semigroup element.*

Proof. Fix $D \geq 1$. We will prove that the result holds for all fat trees Δ with at most D vertices. If Δ consists only of a single vertex, then it equals the squarefree divisor complex of $2(D+1) \in \langle 1 \rangle$. From here, we proceed by induction on the number of vertices of Δ. Let $F_1, F_2, \ldots, F_k$ denote the facets of Δ, let Δ' denote the simplicial complex with facets $F_1, \ldots, F_{k-1}$, and suppose $F = F_k \cap (F_1 \cup \cdots \cup F_{k-1})$ is a simplex in Δ'. Writing $F_k \setminus (F_1 \cup \cdots \cup F_{k-1}) = \{v_1, \ldots, v_r\}$, and consider gluing F_k onto F as a sequence of gluings one vertex at a time, by first gluing $F \cup \{v_1\}$ onto F, then $F \cup \{v_1, v_2\}$ onto $F \cup \{v_1\}$, and so forth. This allows us to assume $|F_k| = |F| + 1$.

For the induction, assume $\Delta' = \Delta_m^S$ for some numerical semigroup $S = \langle n_1, \ldots, n_d \rangle$. Furthermore, we will assume (and prove at each subsequent step) that the following conditions hold:

(i) $n_1 > n_2 > \cdots > n_d$;
(ii) $m > 2n_1 + \cdots + 2n_d$; and
(iii) any sum of distinct generators in S is uniquely factorable (in particular, this implies $\Delta_{n_G} = 2^G$ for any $G \subset [d]$), and

Fixing any prime $p > b = m - n_F$, we claim $\Delta = \Delta_{pm}^T$ with $T = pS + b\langle 1 \rangle$ by Theorem 2.6. Indeed, $F \in \Delta_m^S$ implies $b \in S$, $b \neq n_1, \ldots, n_d$ follows from assumption (ii), and assumption (iii) imples the last hypothesis of Theorem 2.6.

It remains to verify assumptions (i) through (iii) hold for the next round of induction. Since $p > b$, we know $b < n_d$, which verifies (i). Moreover, since n_1 equals the product of all primes at previous steps, assumption (i) from the previous round implies

$$2pn_1 + \cdots + 2pn_d + 2b \leq 2dpn_1 < 2(D+1)pn_1 = pm,$$

so (ii) is satisfied. Lastly, we observe that any sum of distinct generators of T including b equals b modulo p, and thus is also uniquely factorable, ensuring (iii). □

Corollary 2.8. *Any fat forest occurs as the squarefree divisor complex of some numerical semigroup element.*

Proof. Theorem 2.7 implies any fat tree can be realized, and by choosing different primes in Theorem 2.6, their disjoint union can be realized by Theorem 2.4. □

3. Elements with nonzero Euler characteristic

In this section, we turn our attention to particular classes of numerical semigroups.

3.1. **Supersymmetric numerical semigroups.** Suppose $t_1, t_2, \ldots, t_d \in \mathbb{Z}_{\geq 2}$ are pairwise coprime, and let $L = t_1 t_2 \cdots t_d$. Numerical semigroups of the form $\langle L/t_1, \ldots, L/t_d \rangle$ are called *supersymmetric* [8]. Equivalently, a numerical semigroup $\langle n_1, n_2, \ldots, n_d \rangle$ is supersymmetric if $L/n_1, \ldots, L/n_d$ are pairwise coprime, where $L = \mathrm{lcm}(n_1, \ldots, n_d)$.

Theorem 3.1. *Suppose $t_1, t_2, \ldots, t_d \in \mathbb{Z}_{\geq 2}$ are pairwise coprime, let $L = t_1 t_2 \cdots t_d$, and let $S = \langle L/t_1, \ldots, L/t_d \rangle$. If $k \in \mathbb{Z}_{\geq 0}$, then Δ_{kL} contains exactly those faces F satisfying $|F| \leq k$. Further, $\chi(\Delta_{kL}) = (-1)^k \binom{d-1}{k}$, which is zero precisely when $k \geq d$.*

Proof. For any face $F \subseteq [d]$ with $|F| \le k$, we have

$$kL = (k - |F|)L + |F|L = (k - |F|)t_1(L/t_1) + \sum_{i \in F} t_i(L/t_i) \in S,$$

meaning $F \in \Delta_{kL}$. On the other hand, suppose $|F| \ge k+1$ and $kL - n_F \in S$, so there is some factorization $kL = \sum_{i=1}^d a_i(L/t_i)$ with $a_i > 0$ for each $i \in F$. For each i, taking the above equation modulo t_i, we conclude $t_i \mid a_i$. As such,

$$kL = \sum_{i=1}^d a_i(L/t_i) \ge \sum_{i \in F} a_i(L/t_i) \ge \sum_{i \in F} t_i(L/t_i) \ge (k+1)L,$$

which is impossible, so we conclude $F \notin \Delta_{kL}$.

The last statement follows from the observation that there are $\binom{d}{i}$ faces with size i and the well-known identity $\sum_{i=0}^k (-1)^i \binom{d}{i} = (-1)^k \binom{d-1}{k}$; see, e.g. [14, p. 165]. □

Theorem 3.1 identifies certain elements of supersymmetric S that have nonzero Euler characteristic. In fact, these are the only such elements.

Theorem 3.2. *Suppose $t_1, t_2, \ldots, t_d \in \mathbb{Z}_{\ge 2}$ are pairwise coprime, let $L = t_1 t_2 \cdots t_d$, and let $S = \langle L/t_1, \ldots, L/t_d \rangle$. If $m \in S$ and $L \nmid m$, then $\chi(\Delta_m) = 0$.*

Proof. Consider two factorizations $m = \sum a_i(L/t_i) = \sum b_i(L/t_i)$. For each i, define $\overline{a}_i$ to be the unique integer in $[0, a_i)$ congruent to a_i modulo t_i, and define $\overline{b}_i$ similarly. We can then write

$$m = qL + \sum \overline{a}_i(L/t_i) = q'L + \sum \overline{b}_i(L/t_i)$$

for some $q, q' \in \mathbb{Z}_{\ge 0}$. For each i, we take the above equality modulo t_i to conclude $\overline{a}_i \equiv \overline{b}_i \bmod t_i$. However, $0 \le \overline{a}_i, \overline{b}_i < t_i$, so $\overline{a}_i = \overline{b}_i$ for all i. Furthermore, $q = q'$.

Now, let $\overline{F} = \{i : \overline{a}_i > 0\}$, which must be nonempty since $L \nmid m$, and fix $j \in \overline{F}$. By the above argument, every factorization $m = \sum a_i(L/t_i)$ has $a_j > 0$, meaning if $G \subset [d]$ with $j \notin G$, then $G \in \Delta_m$ if and only if $G \cup \{j\} \in \Delta_m$. As such, we conclude $\chi(\Delta_m) = 0$ since G and $G \cup \{j\}$ have opposite signs in the formula for $\chi(\Delta_m)$. □

Remark 3.3. Since supersymmetric numerical semigroups are precisely those with a unique Betti element [13], the expression for their Hilbert series resulting from Theorems 3.1 and 3.2 can be more directly derived.

3.2. **Embedding dimension 3 numerical semigroups.** We conclude by classifying the squarefree divisor complexes with nonzero Euler characteristic in any minimally 3-generated numerical semigroup. Note that some of the results in this section also appeared in [3].

Theorem 3.4. *Fix a numerical semigroup $S = \langle n_1, \ldots, n_d \rangle$. For any element $m \in S$, we have $\Delta_m = 2^{[d]} \setminus \{[d]\}$ if and only if $m - n_{[d]} \in \mathrm{PF}(S)$.*

Proof. First, suppose $m - n_{[d]} \in \mathrm{PF}(S)$. This means $m - n_{[d]} \notin S$, but $m - n_F \in S$ for each proper subset $F \subsetneq [d]$, so each such $F \in \Delta_m$. Conversely, suppose $\Delta_m = 2^{[d]} \setminus \{[d]\}$. Since $[d] \notin \Delta_m$, $m - n_{[d]} \notin S$, but given any positive $s \in S$, writing $s = n_j + s'$ for some $s' \in S$ and some j, and letting $F = [d] \setminus \{j\}$, we have

$$m - n_{[d]} + s = m - (n_{[d]} - n_j) + s' = m - n_F + s' \in S,$$

which implies $m - n_{[d]} \in \mathrm{PF}(S)$. □

We combine the above result with the following known result to prove Theorem 3.6.

Theorem 3.5 ([19, Example 7.23])**.** *For a numerical semigroup $S = \langle n_1, n_2, n_3 \rangle$, let*

$$m_i = \min\{m \in \mathbb{Z}_{\geq 0} : n_i \mid m \text{ and } m \in \langle n_j, n_k \rangle\}$$

for each $i \in \{1, 2, 3\}$, where $\{i, j, k\} = \{1, 2, 3\}$. The elements $m_1, m_2, m_3 \in S$ are the only elements of S with disconnected squarefree divisor complexes.

Theorem 3.6. *Fix a numerical semigroup $S = \langle n_1, n_2, n_3 \rangle$, and define m_1, m_2, and m_3 as above. If $m \in S$, then $\chi(\Delta_m) \neq 0$ if and only if $m \in \{0, m_1, m_2, m_3\} \cup (n_{[3]} + \mathrm{PF}(S))$. In particular, at most 6 elements $m \in S$ have $\chi(\Delta_m) \neq 0$.*

Proof. Up to symmetry, the only simplicial complexes on $\{1, 2, 3\}$ with nonzero Euler characteristic are those with facet sets

$$\{\emptyset\}, \quad \{\{1\}, \{2\}\}, \quad \{\{1\}, \{2\}, \{3\}\}, \quad \{\{1, 2\}, \{3\}\}, \quad \text{and} \quad \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}.$$

Of these, the first only occurs as Δ_0, Theorem 3.4 implies the last complex coincides with Δ_m precisely when $m - n_{[3]} \in \mathrm{PF}(S)$, and the remaining complexes are disconnected, which can only occur for some m_i by Theorem 3.5. The final claim follows from the fact that $|\mathrm{PF}(S)| \leq 2$ by [19, Corollary 9.22]. □

Remark 3.7. Let $n_1, n_2 \in \mathbb{Z}_{\geq 2}$ with $\gcd(n_1, n_2) = 1$, let

$$n_3 = n_1 n_2 - n_1 - n_2 = \mathrm{F}(\langle n_1, n_2 \rangle),$$

and let $S = \langle n_1, n_2, n_3 \rangle$. This family of 3-generated numerical semigroups arises in constructing the Frobenius variety [18], and its factorization properties are of interest [6]. Using Theorem 3.6, we compute

$$B = \{0, n_2 + n_3, n_1 + n_3, 2n_3, n_1 + 2n_3, n_2 + 2n_3\}.$$

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