Bifurcus Semigroups and Rings*

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A bifurcus semigroup or ring is defined as possessing the strong property that every nonzero nonunit nonatom may be factored into two atoms. We develop basic properties of such objects as well as their relationships to well-known semigroups and rings.
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1. INTRODUCTION AND BASIC PROPERTIES

Factorization theory has traditionally considered unique factorization as "good", and focused attention on semigroups whose factorization is close (in various precise ways) to this ideal. We instead propose to consider semigroups that are by some measures the "worst", those whose factorization is as far as possible from this ideal.

Definition 1.1. We call an atomic semigroup *bifurcus* if every nonunit nonatom can be factored into two atoms. We call a ring *bifurcus* if every nonzero nonunit nonatom can be factored into two atoms. We call a bifurcus semigroup or ring *nontrivial* if it contains at least one (nonzero) nonunit.

We do not require our semigroups to be commutative, cancellative, or possess bounded factorization. We also do not require our rings to possess multiplicative identities. In the sequel, we develop various properties of bifurcus semigroups and rings and give some examples. For a background to factorization theory, please see [5]. For additional undefined terms, please see [2, 3].

We begin by presenting some basic properties and calculating standard factorization invariants for bifurcus semigroups.

Theorem 1.1. *Let S be a nontrivial bifurcus semigroup, and let x be a (nonzero) nonunit nonatom in S. Then:*

1.If S is either left or right cancellative, then S contains infinitely many atoms, and the divisor closure $[[x]]$ is not finitely generated.
2.S contains no strong atoms.
3.Let $\mathfrak{L}(x)$ denote the set of factorization lengths of x, and let $L(x) = \sup \mathfrak{L}(x)$. Then $\mathfrak{L}(x)$ is the set of integers in $[2, L(x)]$.
4.The elasticity $\rho(x)$ satisfies $2\rho(x) \in \mathbb{N} \cup \{\infty\}$.
5.The elasticity $\rho(S) = \infty$.
6.The delta set $\Delta(S) = \{1\}$.
7.The catenary degree $c(S) = 3$.
8.The tame degree $t(S) = \infty$.
9.The critical length of S is 3.

Proof.

1. For each $n \in \mathbb{N}$, we write $x^n = a_n b_n$, where a_i, b_i are atoms. Suppose that S is left cancellative (right cancellative is similar). We will show that $\{a_1, a_2, \ldots\}$ are distinct. Otherwise we have $a_i = a_{i+j}$ for some $i, j \in \mathbb{N}$. We

have $a_ib_ix^j = x^ix^j = x^{i+j} = a_{i+j}b_{i+j} = a_ib_{i+j}$. Applying left cancellation yields a factorization of the atom b_{i+j} into nonunits, a contradiction. The second statement holds since $\{a_1, a_2, \ldots\} \subseteq [[x]]$.

2. Let y be a strong atom. Applying the bifurcus property we have $y^3 = ab$, for atoms a, b. Applying the strong atom property, $a = \epsilon y^\alpha, b = \epsilon' y^\beta$, with $\alpha + \beta = 3$ and ϵ, ϵ' units. Without loss of generality $\alpha \geq 2$; but then a is not an atom.

3. It suffices to prove that if $m \in \mathfrak{L}(x)$ with $m \geq 3$, then $m - 1 \in \mathfrak{L}(x)$. Let $x = b_1b_2b_3 \cdots b_m$ be a factorization. By the bifurcus property $b_1b_2b_3 = cd$, hence $x = cdb_4 \cdots b_m$, a factorization of length $m - 1$.

4. Follows directly from the bifurcus property, which gives $\min \mathfrak{L}(x) = 2$.

5. Consider a^n for any atom a, as $n \to \infty$. sup $\mathfrak{L}(a^n) \geq n$, but inf $\mathfrak{L}(a^n) = 2$.

6. Follows from property (3).

7. Given any factorization of x, we iteratively apply the construction from (3) to get a sequence of factorizations, each of distance three from each other, ending in two atoms. Given two factorizations, we apply the preceding process twice to get $f_1 \to f_2 \to \cdots \to ab$ and $g_1 \to g_2 \to \cdots \to cd$. Reversing and combining, we have a sequence of factorizations $f_1 \to f_2 \to \cdots \to ab \to cd \to \cdots \to g_2 \to g_1$, each of distance at most three.

8. Follows from (5) and Thm 1.6.6 from [5] that gives $\rho(S) \leq t(S)$.

9. Immediately follows from the observation that $\min \mathfrak{L}(x) = 2$. In fact, the bifurcus property characterizes monoids with critical length 3.

▮

The bifurcus property excludes many classes of well-studied and familiar semigroups and rings.

THEOREM 1.2. *The following are not bifurcus:*

1.Block monoids $B(G_0)$, for $G_0 \subseteq G$.
2.Krull monoids.
3.Rings of integers of any algebraic number field.
4.Diophantine monoids.
5.Numerical semigroups.

Proof.

1. Let $x \in B(G_0)$ be a nonunit nonatom. We may express x as the multiset $\{x_1^{m_1}, x_2^{m_2}, \ldots, x_k^{m_k}\}$ where $k \geq 1$, $x_i \in G_0$, and $m_i \geq 1$ (for all $i \in [1, k]$). Consider the function γ from multisets drawn from $\{x_1, x_2, \ldots, x_k\}$

to $\mathbb{N}_0^k$ that gives the multiplicity of each element (e.g. $\gamma(x) = (m_1, m_2, \ldots, m_k)$). For all $n \in \mathbb{N}$, the bifurcus property gives a factorization of $x^n = a(n)b(n)$ into two atoms. Without loss we may assume that $\gamma(a(n))_1 \geq \gamma(b(n))_1$. Note that $\gamma(a(n)) + \gamma(b(n)) = \gamma(x^n) = (nm_1, nm_2, \ldots, nm_k)$, hence $\gamma(a(n))_1 \in [\frac{nm_1}{2}, nm_1]$. Consider now the set $S = \{\gamma(a(1)), \gamma(a(2)), \ldots\}$, a subset of $\mathbb{N}_0^k$. Because of the condition on the first coordinates of the elements of S, $|S| = \infty$. By a classical theorem attributed to Lothaire (in [7]) or Dickson (in [5]), $\mathbb{N}_0^k$ has no infinite antichain in the usual partial ordering. Hence there must be some i, j with $\gamma(a(i)) \geq \gamma(a(j))$; but then $a(j)|a(i)$ in $B(G_0)$, and hence $a(i)$ is not an atom, contrary to assumption.

2. Follows from (1) and Thm 2.5.8 from [5], which states that all reduced Krull monoids are block monoids.

3. Follows from Thm 1.7.3 from [5] together with Theorem 1.1 (5).

4. Follows from (2) and, e.g., [4], which shows that Diophantine monoids are Krull.

5. Follows from, e.g., [6], which shows that numerical semigroups are cancellative and must have a finite number of atoms. This is violative of Theorem 1.1 (1).

∎

While it may seem that the bifurcus property is rare, the following result shows that in fact every semigroup can be embedded into a bifurcus semigroup.

Theorem 1.3. *Let R be any semigroup. Let S be any atomic semigroup with no units. Then $T = R \times S \times S$ is bifurcus.*

Proof. Note that T has no units since S does not. Let $x \in T$ be a nonatom. We factor $x = yz$, where $y = (r_y, u_y, v_y), z = (r_z, u_z, v_z)$, and $x = (r_y r_z, u_y u_z, v_y v_z)$. We write $u_y u_z = pu$ and $v_y v_z = vq$, for some atoms $p, q \in S$. Set $y' = (r_y, p, v), z' = (r_z, u, q)$. These are atoms in T, and $x = y'z'$. ∎

2. EXAMPLES

We provide several examples of bifurcus rings. These do not have multiplicative units, but (1) and (3) do not have zero divisors. The conditions imposed on m, n are all necessary – if n is a prime power, then neither (1) nor (3) is bifurcus; if $m = 1$, then (2) is not bifurcus.

Example 2.1. The following are bifurcus rings:

1. $n\mathbb{Z}$, for n not a prime power.

2. $(m\mathbb{Z}) \times (n\mathbb{Z})$ for m, n natural numbers greater than 1.
3. The subring of $n \times n$ matrices consisting of matrices with all entries identical integers, for n not a prime power.

Proof.

1. Atoms in our ring are nx where $n \nmid x$. Write $n = pqr$ where p, q are prime and might divide $r \in \mathbb{Z}$. Consider nonatom $z = (nx)(ny) = p^a q^b r^2 s$, where $p, q \nmid s$. Note that $a, b \geq 2$; hence we can factor $z = (pqr(q^{b-2}s))(pqr(p^{a-2}))$. These are atoms since $n \nmid q^{b-2}s$ and $n \nmid p^{a-2}$.
2. Consider nonatom $z = (ma, nb) \times (mc, nd) = (m, nbd) \times (mac, n)$, a factorization into two atoms.
3. This ring is isomorphic with $\mathbb{Z}$, with the usual addition but with multiplication given by $x \star y = nxy$. Atoms are those integers that are not multiples of n. Write $n = pqr$ where p, q are prime and might divide $r \in \mathbb{Z}$. Consider nonatom $z = x \star y = nxy = np^a q^b s$ where $p, q \nmid s$. Set $x' = p^a, y' = q^b s$. These are atoms and $z = x' \star y'$.
∎

Bifurcus semigroups turn out to be common among (non-commutative) matrix semigroups (see [1]). We give just one example.

Example 2.2. Let $n > 1$ and let S denote the semigroup of $n \times n$ rank one matrices with entries from $\mathbb{N}$. Then S is bifurcus.

Proof. Let gcd denote the usual greatest common divisor function, which we will apply to the entries of matrices and vectors. Recall that $a \in S$ may be expressed (non-uniquely) as $a = uv^T$, for u, v column n-vectors. We claim that $\gcd(a) = \gcd(u)\gcd(v)$. We assume without loss that $\gcd(u) = \gcd(v) = 1$ and prove $\gcd(a) = 1$ by instead considering $\frac{a}{\gcd(u)\gcd(v)} = \frac{u}{\gcd(u)} \frac{v^T}{\gcd(v)}$. If $p | \gcd(a)$, then p divides each entry of the i^{th} columns of a, namely $v_i u$. Since p cannot divide each entry of u, $p | v_i$. But this holds for each i, hence p divides each entry of v, a contradiction. Hence $\gcd(a) = 1$, as desired.

Now consider nonatom $a = bc = (u_b v_b^T)(u_c v_c^T) = (v_b^T u_c)(u_b v_c^T)$. Note in passing that since all entries are from $\mathbb{N}$, $\gcd(a) \geq v_b^T u_c \geq n$ for every nonatom a. Set $a' = \frac{a}{\gcd(a)} = u' v'^T$; by the previous claim $\gcd(u') = \gcd(v') = 1$. Set $x = [\gcd(a) - n + 1, 1, \ldots, 1]^T, y = [1, 1, \ldots, 1]^T$. We have $a = \gcd(a) a' = (x^T y)(u' v'^T) = (u' x^T)(y v'^T)$. Because $\gcd(u') = \gcd(v') = \gcd(x) = \gcd(y) = 1$, by the previous claim $\gcd(u' x^T) = \gcd(y v'^T) = 1$, and since $n > 1$ these are both atoms. ∎

We conclude with some unanswered questions.

Open Problems 2.1.

1. Does there exist a bifurcus ring with 1? A bifurcus domain?

2. Can every ring/domain be embedded in a bifurcus ring?

3. Can a bifurcus semigroup posess finitely many atoms?

Note that by Theorem 1.1(1), such an example would be neither left nor right cancellative. Further, such an example must be finite (since N atoms yields at most N^2 ordered pairs of atoms), and therefore must not possess bounded factorization.

4. Can a bifurcus semigroup be inside factorial or Cale?

5. Can a bifurcus semigroup be locally tame?

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