Membership and Elasticity in Certain Affine Monoids

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http://vadim.sdsu.edu/2019-Hawaii-talk.pdf

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Shameless advertising

Please encourage your students to apply to the
San Diego State University Mathematics REU
(for next summer).

Projects in Nonunique Factorization; summer 2019 projects in numerical semigroups.

`http://www.sci.sdsu.edu/math-reu/index.html`

This work was done jointly with Jackson Autry.

Affine Monoids: definition

For us, an affine monoid is a set S, satisfying:

- $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} \subseteq S \subseteq \mathbb{N}_0^2$
- S is closed under $+$

Given $\{t_1, t_2, \ldots, t_k\} \subseteq S$, we define submonoid $\langle t_1, t_2, \ldots, t_k \rangle = \{\sum_{i=1}^{k} \alpha_i t_i : \alpha_i \in \mathbb{N}_0\} \subseteq S$

We further assume that S has embedding dimension 2 or 3, to be defined next.

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Affine Monoids: irreducibles, embedding dimension

A nonzero element $t \in S$ is irreducible if:
there are no nonzero $t_1, t_2 \in S$ with $t = t_1 + t_2$

There is a unique set of irreducibles $\{u, v, \ldots, w\}$ with $S = \langle u, v, \ldots, w \rangle$.
We call $|\{u, v, \ldots, w\}|$ the embedding dimension of S.

We assume that the embedding dimension is 2 or 3; i.e. $S = \langle u, v \rangle$ or $S = \langle u, v, w \rangle$

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Affine Monoids, factorization

Set $S = \langle u, v, w \rangle$ and consider the map:

- $\pi : \mathbb{N}_0^3 \to S$ given by $\pi : (\alpha, \beta, \gamma) \mapsto \alpha u + \beta v + \gamma w$

If $\pi(\alpha, \beta, \gamma) = s$, we call (α, β, γ) a factorization of s. We call π the factorization homomorphism of S.

For $s \in S$, set $Z(s)$ to be the set of all factorizations of s:

- $Z(s) = \pi^{-1}(S)$.

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Affine Monoids, factorization lengths

For $s \in S$ and for $u = (\alpha, \beta, \gamma) \in Z(s)$, define the length of u as:

- $|u| = \alpha + \beta + \gamma$.

For $s \in S$, define the set of lengths of s as:

- $L(s) = \{|u| : u \in Z(s)\}$.

For $s \in S$, define the elasticity of s as:

- $\rho(s) = \frac{\max L(s)}{\min L(s)}$

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What's This Talk All About?

Our results are addressing two questions
(each for embedding dimension $2, 3$):

Membership Problem:
Given affine monoid S and $x \in \mathbb{N}_0^2$, is $x \in S$?

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Classical Tool 1: SNF and Determinantal Divisors

- Smith Normal Form:

  Given 2×3 matrix M, with integer entries.

  There must exist square unimodular matrices U, V, with:

  $UMV = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_1 d_2 & 0 \end{bmatrix}$

  d_i called determinantal divisors of M.

  d_i is the gcd of all the $i \times i$ minors of M.

  In particular, $d_1 = \gcd(M)$.

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  Multiplying M on either side by a unimodular matrix, leaves determinantal divisors unchanged.

  Set $u = Mv$, for any $v \in \mathbb{Z}^2$. The determinantal divisors of $[M|u]$ are the same as that for M.

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Classical Tool 2: MHNF

- Modified Hermite Normal Form

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New Tool: ϕ

Set $\mathbb{Q}^\star = \mathbb{Q}^{\geq 0} \cup \{\infty\}$.

Define $\phi : \mathbb{N}_0^2 \to \mathbb{Q}^\star$ via $\phi : \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \mapsto \frac{a}{b}$ (∞ if $b = 0$)

ϕ will largely answer our questions.

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Properties of ϕ

Thm: Let $u, v \in \mathbb{N}_0^2$. Then $\phi(u+v) \in [\phi(u), \phi(v)]$.
Note: This interval is understood to be $[\phi(v), \phi(u)]$ if $\phi(v) < \phi(u)$.

Cor: Let $u, v \in \mathbb{N}_0^2$, and $s \in \langle u, v \rangle$. Then $\phi(s) \in [\phi(u), \phi(v)]$.

Cor: Let $u, v \in \mathbb{N}_0^2$, and $s \in \langle u, v \rangle$. Let U be unimodular 2×2. Then $Us \in \langle Uu, Uv \rangle$ and $\phi(Us) \in [\phi(Uu), \phi(Uv)]$.

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Embedding Dimension 2

Set $S = \langle \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \rangle$, and $s = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$.

Note that $d_2(\left[\begin{smallmatrix} 0 & a \\ 1 & b \end{smallmatrix}\right]) = a$.

If $s \in S$, then both:

- $\phi(s) \in [\phi(\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]), \phi(\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right])] = [0, \frac{a}{b}]$; and
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Also, $\rho(s) = 1$, new proof.

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Set $S = \langle \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right], \left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right] \rangle$, and $s = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$, where we assume that $\phi(\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]) < \phi(\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]) < \phi(\left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right])$.

Note that $d_2(\left[\begin{smallmatrix} 0 & a & c \\ 1 & b & d \end{smallmatrix}\right]) = \gcd(a, c, bc - ad) = \gcd(a, c)$.

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Still not enough for sufficiency!

Embedding Dimension 3, intermezzo

Example: $S = \langle \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 11 \\ 10 \end{bmatrix}, \begin{bmatrix} 10 \\ 3 \end{bmatrix} \rangle$, $s = \begin{bmatrix} 199 \\ 119 \end{bmatrix}$.

$\phi(s) \in [0, \frac{10}{3}]$

$199 \in \langle 11, 10 \rangle$ (uniquely)

$d_2(\begin{bmatrix} 0 & 11 & 10 \\ 1 & 10 & 3 \end{bmatrix}) = d_2(\begin{bmatrix} 0 & 11 & 10 & 199 \\ 1 & 10 & 3 & 119 \end{bmatrix}) = 1$

But still $s \notin S$.

Embedding Dimension 3, conclusion

Set $S = \langle \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}a\\b\end{smallmatrix}\right], \left[\begin{smallmatrix}c\\d\end{smallmatrix}\right] \rangle$, and $s = \left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]$, where we assume that $\phi(\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]) < \phi(\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]) < \phi(\left[\begin{smallmatrix}c\\d\end{smallmatrix}\right])$. Assume $bc - ad = 1$.

If $s \in S$, then:

- $\phi(s) \in [\phi(\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]), \phi(\left[\begin{smallmatrix}c\\d\end{smallmatrix}\right])] = [0, \frac{c}{d}]$; and
- $x \in \langle a, c \rangle$

Thm: These necessary conditions are also sufficient.

If $ad - bc \neq 1$, all still open.

Embedding Dimension 3, conclusion

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Defining p, q, r

Set $S = \langle \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right], \left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right] \rangle$, and $s = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$, where we assume that $bc - ad = 1$. (implies $\frac{a}{b} < \frac{c}{d}$)

Suppose that $x \in \langle a, c \rangle$. There are unique choices of $q, r \in \mathbb{N}_0$ such that $x = qa + rc$ and $0 \leq q < c$.

Suppose that $s \in S$. Then there is a unique choice of $p \in \mathbb{N}_0$ such that $y = p + qb + rd$, i.e. $s = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = p\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] + q\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] + r\left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right]$.

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Elasticity in Embedding Dimension 3

Set $S = \langle [\begin{smallmatrix}0\\1\end{smallmatrix}], [\begin{smallmatrix}a\\b\end{smallmatrix}], [\begin{smallmatrix}c\\d\end{smallmatrix}] \rangle$, and $s = [\begin{smallmatrix}x\\y\end{smallmatrix}]$, with $bc - ad = 1$. Let $p, q, r \in \mathbb{N}_0$ satisfy $s = p[\begin{smallmatrix}0\\1\end{smallmatrix}] + q[\begin{smallmatrix}a\\b\end{smallmatrix}] + r[\begin{smallmatrix}c\\d\end{smallmatrix}]$ with $0 \leq q < c$.

Thm 1: If $\frac{x}{y} \leq \frac{a}{b}$, then the min/max factorizations of s have lengths $p + q + r$ and $p + q + r + \lfloor \frac{r}{a} \rfloor (c - a - 1)$.

Note: $c - a - 1$ could be positive, zero, negative.

Thm 2: If $\frac{x}{y} \geq \frac{a}{b}$, then the min/max factorizations of s have lengths $p + q + r$ and $p + q + r + p(c - a - 1)$.

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Elasticity Limits

Set $S = \langle \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}a\\b\end{smallmatrix}\right], \left[\begin{smallmatrix}c\\d\end{smallmatrix}\right] \rangle$, and $s = \left[\begin{smallmatrix}x\\y\end{smallmatrix}\right] \in S$, with $bc - ad = 1$.

We expect $\phi(s)$ largely determines elasticity.

$\phi(ks) = \phi(s)$ for all $k \in \mathbb{N}$.

Thm: Set $\tau = \mathrm{sign}(c - a - 1)$. Then

$$\lim_{k\to\infty} \rho(ks) = \begin{cases} \left(\frac{c}{a}\frac{a-\frac{x}{y}(b-1)}{c-\frac{x}{y}(d-1)}\right)^{\tau} & \frac{x}{y} \le \frac{a}{b} \\ \left(c\frac{(c-a)-\frac{x}{y}(d-b)}{c-\frac{x}{y}(d-1)}\right)^{\tau} & \frac{x}{y} \ge \frac{a}{b} \end{cases}$$

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For Further Reading

Membership and Elasticity in Certain affine Monoids
https://vadim.sdsu.edu/ap3.pdf