MATH601 Spring 2008
Handout 9: Rifts in Mathematics
Unit 4: Cardinals

Axiom of Choice "AC" (Zermelo [ca. 1904])
Given any set of nonempty sets, we can choose exactly one element from each of the nonempty sets.

"The Axiom of Choice is necessary to select a set from an infinite number of socks, but not an infinite number of shoes." – Bertrand Russell

Equivalent to:

- Any two cardinals are comparable. In other words, for any sets S, T, either $|S| \leq |T|$ or $|T| \leq |S|$.
- For any sets S, T, if either is infinite, then $|S| \times |T| = |S| + |T| = \max\{|S|, |T|\}$.
- Every set can be well-ordered.
- Every vector space has a basis.
- Zorn's lemma: Every poset where every chain has an upper bound contains a maximal element.
- Tychonoff's theorem: Every product of compact topological spaces is compact.

Kurt Gödel in 1940 proved that AC is not false. In 1963 Paul Cohen proved that AC is not true.

Yes:

- There are non-measurable sets.
- Every field has an algebraic closure.
- The Law of the Excluded Middle: In any specific context, statements must be either true or false.
- Banach-Tarski paradox: A three-dimensional solid sphere can be divided into finitely many pieces which can be rearranged to form two solid spheres, each of the same size as the original.

No:

- Some vector spaces have have multiple bases, of different cardinalities.
- There is a function $f : \mathbb{R} \to \mathbb{R}$ such that f is not continuous at a, but for any sequence $\{x_n\} \to a$, $\lim_{n\to\infty} f(x_n) = f(a)$.
- GCH is false (see below).

Continuum Hypothesis "CH" (Cantor [ca. 1890])
There is no set whose cardinality is strictly between $|\mathbb{Z}|$ and $|\mathbb{R}|$.

Generalized Continuum Hypothesis "GCH"
For any infinite set S, there is no set whose cardinality is strictly between $|S|$ and $|2^S|$.

Kurt Gödel in 1940 proved that CH and GCH are not false. In 1963 Paul Cohen proved that CH and GCH are not true.

Constructivist Mathematics

No "there exists" without an explicit construction, No Law of Excluded Middle, no proof by contradiction, no axiom of choice, no Intermediate Value Theorem.