MATH601 Spring 2008 Handout 6
Unit 3: Hyperreal Numbers

In this unit, we encounter the hyperreals, an extension of the reals. A calculus text using hyperreals can be found at: `http://www.math.wisc.edu/~keisler/calc.html`. Please download the first two chapters. You might also be interested in the appendix, which gives solutions to some exercises, and the epilogue, which builds the hyperreals up from scratch (we will not explore this). Another historical perspective can be found at: `http://mathforum.org/dr.math/faq/analysis_hyperreals.html`.

The hyperreals $\mathbb{R}^\star$ (sometimes ${}^\star\mathbb{R}$) are an ordered, dense, non-Archimedean field, that extend the reals $\mathbb{R}$. The major issue we explored in Unit 2 was extending the rationals $\mathbb{Q}$ to the reals; this will *not* be an issue in this unit, because of the following two principles, which we take as given:

Extension Principle. This has three parts:

- $\mathbb{R} \subseteq \mathbb{R}^\star$, and they share the same ordering $\leq$.
- $\mathbb{R} \neq \mathbb{R}^\star$; in fact, $\mathbb{R}^\star$ contains nonreal infinitesimal and (nonreal) infinite elements.
- Every real function f can be extended to a hyperreal function $f^\star$, that agrees with f on $\mathbb{R}$.

Transfer Principle.
Every finite real statement that holds for a real function f is also true for its hyperreal extension $f^\star$.

The transfer principle is very powerful. For example, it proves that $\mathbb{R}^\star$ is an ordered, dense, field, because each of the required properties get transferred from the reals to the hyperreals. However, we cannot write "$\mathbb{R}$ is Archimedean" using finitely many real statements; this statement is written as "for all numbers a, either $a < 1$ or $a < 1+1$ or $a < 1+1+1$ or $\ldots$", which is infinitely many real statements. So, this property of reals does not automatically transfer, and in fact is not true for hyperreals.

The infinitesimals are the set of numbers that are smaller than every positive real, and greater than every negative real. They are all "infinitely close" to 0. In $\mathbb{R}$, there is only one infinitesimal, namely 0. In $\mathbb{R}^\star$, there are lots of infinitesimals: 0, but also positive and negative infinitesimals. Think of the infinitesimals as a microscopic cloud that surrounds 0, a cloud smaller in diameter than every positive real number.

By applying the transfer principle, we get a lot more. If we add 7 to every infinitesimal, we discover a similar cloud surrounding 7; in fact, every real number has its own cloud surrounding it. If we take the reciprocal of a positive infinitesimal, we must get a positive number; further, this reciprocal must be larger than every positive real number, since the infinitesimal was smaller than every positive real number. Hence we get infinite numbers, both positive and negative. Applying the transfer principle repeatedly we can learn the arithmetic of $\mathbb{R}^\star$.

To practice this arithmetic on $\mathbb{R}^\star$, for Monday please read sections 1.4, 1.5, and do exercises $5, 15, 25, 35, 40, 41, 42, 43, 44$ from section 1.5.

Abraham Robinson 1918-1974
UCLA, Yale
Developed model theory, nonstandard analysis

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Wrote textbook whose excerpt we're using:
`http://www.math.wisc.edu/~keisler/calc.html`
(we're focusing on §§1.4, 1.5, 1.6, 2.1)