Math 579: Combinatorics

Spring 2026 Coursepack

Syllabus Highlights

Prerequisites:
Students are expected to be very comfortable with algebraic calculations, and at least somewhat comfortable with proofs. Math 245 and Math 254 are the course prerequisites.

Course Structure:
This course has in-person meetings twice weekly, all of which students must attend. These meetings will be split into a first half (approx. 30 mins), followed by a 5 minute break, followed by the remaining time (approx. 40 mins). The first half will consist of student presentations, while the second half will consist of students working in groups on the unit exercises, with the instructor offering help and guidance as needed.

Grading:
Unit exams, presentations, and the final exam, are all normalized to lie between 50% (blank but present) and 100% (perfect score). Missing grades are still worth 0%. The cutoffs for each letter grade are as below. Grade appeals are due one week from when it's posted to Canvas.

What?	When?	Why?
Student Presentations	various	50
Attendance	throughout	50
Unit 1 Exam	Thu. Feb. 5	100
Unit 2 Exam	Thu. Mar. 19	100
Unit 3 Half-Exam	Thu. Apr. 16	50
Team Exam Score	various	50
Spring Break	Week of Mar.31, Apr.2	no class
Final Exam	Thu. May 7 10:30-12:30	200
Total		600

A: 552 A-: 540 B+: 528 B: 492 B-: 480 C+: 468
C: 432 C-: 420 lol D+: 408 D: 372 F: 0

Attendance Scores:
Student attendance scores begin at 50/50 and can only decrease. Students may miss one class without penalty. After that, each unexcused absence incurs a 5 point penalty to participation. Coming late, leaving early, or being disengaged (e.g. on your phone) will also lead to penalties.

Teams:
Students will divide themselves into four teams, with whom they will work closely all semester. The end of the day on February 10 is the Team Adjustment Deadline. At this time your team will be locked in for the rest of the semester. Students have been pre-placed into Magenta Group, Plum Group, and Violet Group. Teams must include 1-2 people from each group.

To encourage team collaboration, 50 points of your final grade will be the average of all the exam scores of everyone on your team (including you). The instructor will update this score after each exam to help you estimate your current grade.

Discord:
Please join the course Discord, you will likely find it quite helpful: `https://discord.gg/3gXW3HN7GA`

Student Presentations:
The first half of each class will be filled with four student presentations – you will be presenting approximately six times during the semester. All presenters will write their solution on the board simultaneously (before class if possible), and will take turns discussing their solution and answering questions. They will sign up on the signup sheet provided (link in Canvas and in the Discord). Presentations are graded based on correctness as well as discussion accuracy. Your presentation grade is your average score on presentations, scaled up to 50 points.

Exams:
Exams are taken with no access to notes, calculators, phones, smartwatches, or other aids, apart from the coursepack which you will need. Unlimited paper will be provided. Most exam questions will be similar to the exercises. Unit 3 is shorter, so its exam will be shorter. The final exam will be similar to the unit exams in structure, except there will be extra emphasis on Unit 4 (which does not have its own unit exam).

Attendance:
Students are expected to attend every class, paying attention, participating, and taking notes as appropriate. Makeup exams are not given under any circumstances. Students who will miss class due to an official university event or activity (such as athletics), must notify the instructor during the first two weeks of classes. Absences for official university events must be documented with a memorandum from the event's sponsor with that same deadline. Students missing class due to a medical emergency must provide a signed medical excuse justifying the absence. Student Health Services does not provide these.

Academic Integrity:
Students are strongly encouraged to study with their team, and to work together on exercises. Teams work best when everybody contributes. Don't sit passively – jump in there and take control of your education. Similarly, don't let others on your team sit passively – if they later bomb their exams, it will hurt both their grade and yours.

Use of AI to help with homework is discouraged but not forbidden – genAI is like a well-spoken but dumb friend, not a math expert. Trusting an AI solution without understanding is just as foolish as trusting a friend's solution without understanding.

Collaboration on exams is forbidden. Use of written, digital, or online resources during exams is forbidden, except for the coursepack. All violations will be reported to the Center for Student Rights and Responsibilities and will also result in grade reductions or worse. Courses failed due to integrity violations are ineligible for course forgiveness. See SDSU's full policy[1] on academic honesty, or ask the instructor, if you have any doubts or questions.

Do not distribute collections of solutions to people outside your team (hints, parts of problems, or even entire single problems from time to time, are all fine).

Professor:
Vadim Ponomarenko `vponomarenko@sdsu.edu`
Drop-in office hours: Tuesdays and Thursdays 8:30-9:15am and 11am-12:15pm.
Also at other times by appointment. All office hourse held in GMCS 511
Website: `http://vadim.sdsu.edu/`

[1] `https://sacd.sdsu.edu/student-rights/academic-dishonesty/cheating-and-plagiarism`

Unit 1: Elementary Counting and The Twelvefold Way

Sum Principle: If A, B are disjoint sets (i.e. have no elements in common), then $|A \cup B| = |A| + |B|$. Also useful is $|A| = |A \cup B| - |B|$.

Partition Principle: If $\{P_1, \ldots, P_k\}$ is a partition of set S, then $|S| = |P_1| + \cdots + |P_k|$. Further, if $|P_1| = \cdots = |P_k|$, then $|S| = k|P_1|$.

Inclusion-Exclusion Principle: For two sets, $|A \cup B| = |A| + |B| - |A \cap B|$. For three sets, $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$. Similar formulas exist for more sets.

Product Principle: For any[2] sets A, B, consider their Cartesian product $A \times B = \{(a, b) : a \in A, b \in B\}$, a set of two-tuples (also called ordered pairs). Then $|A \times B| = |A| \cdot |B|$.

Pigeonhole Principle: Let A, B be sets with $|A| > |B|$. Then every function $f : A \to B$ is NOT injective.

Generalized Pigeonhole Principle: Let A, B be sets, and $f : A \to B$ a function. Then there is some $b \in B$ such that $|f^{-1}(b)| \geq \left\lceil \frac{|A|}{|B|} \right\rceil$.

The Twelvefold Way: Let $a, b \in \mathbb{N}_0$; let A, B be collections (sets or multisets) with $|A| = a, |B| = b$. We count functions $f : A \to B$, under various restrictions. Are the elements of A different (i.e., A is a set) or identical (i.e., A is a multiset with a single element of some multiplicity)? Are the elements of B different or identical? Do we count all functions, or only injective ones, or only surjective ones?

A	B	All f	Injective f	Surjective f
different	different	b^a	$b^{\underline{a}}$	$b! \left\{ {a \atop b} \right\}$
identical	different	$\left(\!\binom{b}{a}\!\right)$	$\binom{b}{a}$	$\left(\!\binom{b}{a-b}\!\right)$
different	identical	$\sum_{i=1}^{b} \left\{ {a \atop i} \right\}$	$\begin{cases} 1 & \text{if } a \leq b \\ 0 & \text{if } a > b \end{cases}$	$\left\{ {a \atop b} \right\}$
identical	identical	$\sum_{i=1}^{b} p(a, i)$	$\begin{cases} 1 & \text{if } a \leq b \\ 0 & \text{if } a > b \end{cases}$	$p(a, b)$

[2]Throughout this course, all sets will be finite.

Unit 1 Exercises

For Jan. 22: (poker)

A standard deck of cards consists of 52 distinct cards. They are divided into four suits (clubs, diamonds, hearts, spades) of thirteen ranks each (2,3,4,5,6,7,8,9,10,J,Q,K,A). A poker hand consists of a set of five cards, in which the order is as given except A may be chosen to be low or high.

1. The private hand (hole cards) in the game Texas Hold 'Em consists of a set of two cards. How many private hands are possible? Answer in two ways: (i) Use the product principle and the partition principle; and (ii) Use the twelvefold way.

2. The public hand (community cards) in the game Texas Hold 'Em consist of a set of three cards, followed by a single card, then finally another single card. How many public hands are possible?

3. The best two poker hands are straight flush (five consecutive cards all in the same suit) and four of a kind (all four of one rank, plus another card). If all poker hands are equally likely, what are the probabilities of each of those two hands?

4. The next best two poker hands are full house (three cards of one rank, and a pair of some other rank) and flush (five cards of the same suit, but NOT a straight flush). If all poker hands are equally likely, what are the probabilities of each of those two hands?

NOTE: Numbered problems will be solved in class, and then presented. Lettered problems are extra, to work on at home.

a. A straight in poker is five consecutive cards (but NOT a straight flush). If all poker hands are equally likely, what is the probability of a straight?

b. A three-of-a-kind in poker contains three cards of the same rank (but NOT a four of a kind or a full house). If all poker hands are equally likely, what is the probability of a three-of-a-kind?

c. A two-pair in poker contains two pairs, where each pair is of the same rank. It is NOT a four of a kind or a full house. If all poker hands are equally likely, what is the probability of a three-of-a-kind?

For Jan. 27: (words)

Today we will build words (strings) using letters from the ten-letter alphabet $\{A, B, C, D, E,$

5. For $n = 0, 1, 2, 10, 11$, calculate (i) How many words of length n; and (ii) How many words of length n without repeated letters; and (iii) How many words of length n using all ten letters? (a table of Stirling numbers appears at the back)

6. Calculate (i) How many three-letter words that do not include any double letter (e.g., AA or BB or...); and (ii) How many four-letter words that do not include any double letter.

7. Calculate how many four-letter words do not include the specific double letter AA.

8. Calculate how many seven-letter words there are, which are palindromes (read the same backwards and forwards), that do not include any triple letter.

d. Calculate how many eight-letter words contain at least one double letter.

e. Calculate how many eight-letter words there are (on the alphabet $\{A, \ldots, J\}$), which are palindromes (read the same backwards and forwards), that do not include any triple letter.

f. Calculate how many nine-letter words there are (on the alphabet $\{A, \ldots, J\}$), which are palindromes (read the same backwards and forwards), that do not include any triple letter.

<u>For Jan. 29:</u> (pigeonhole)

9. We are given $n+1$ integers from $[2n]$. Prove that there is at least one pair of integers whose sum is $2n+1$.

10. Consider the 987 numbers $n_1 = 3, n_2 = 33, n_3 = 333, \ldots, n_{987} = \underbrace{33\cdots3}_{987}$. Prove that at least one of them must be divisible by 987.

11. In a unit cube one hundred points are given. Prove that some four of them form a tetrahedron whose volume is at most $1/99$.

12. Prove that there are at least thirty people living in San Diego, all born on the same day of the same month of the same year.

g. A sports tournament has $n \geq 2$ participants, by the end of which each player will play one game against each other. Prove that at all times throughout, there must always be at least two players who have completed the same number of games.

h. We are given $n+1$ integers from $[2n]$. Prove that there is at least one pair of integers whose difference is n.

i. Consider the Mersenne numbers $M_i = 2^i - 1$, for $i \geq 1$. Let q be an arbitrary positive odd integer. Prove that some Mersenne number is divisible by q.

j. We are given 175 positive integers, none of which have a prime divisor bigger than 10. Prove that we can find three of them whose product is a perfect cube.

<u>For Feb. 3:</u> (multinomials)

13. Let B be a finite set. Set $b = |B|$, and let $a \in \mathbb{N}_0$. (i) Use the twelvefold way to prove that the number of multisets, whose elements are drawn from B, and whose cardinality is a, is $\left(\!\binom{b}{a}\!\right)$. (ii) We now insist that each element's multiplicity is at least 1 (and the cardinality is still a). Now prove that the number of such multisets is $\left(\!\binom{b}{a-b}\!\right)$.

14. Let $a, b \in \mathbb{N}$. We are given a stars and $b-1$ bars, and are asked to put them in a row (e.g., if $a = b = 4$, we might have $\star\star \mid \star \mid\mid \star$). Prove that the number of ways to do this is $\left(\!\binom{b}{a}\!\right)$. Then prove that the number of ways to do this is $\binom{b+a-1}{b-1}$.
This is called the stars-and-bars proof that $\left(\!\binom{b}{a}\!\right) = \binom{b+a-1}{b-1}$.

15. Let $k, n \in \mathbb{N}$, and consider the equation $x_1 + x_2 + \cdots + x_k = n$. Find the number of solutions to this equation in nonnegative integers.
These are called weak compositions of n.

16. What is the number of compositions of 50 into four odd parts?

k. Let $k, n \in \mathbb{N}$, and consider the equation $x_1 + x_2 + \cdots + x_k = n$. Find the number of solutions to this equation in positive integers.
These are called compositions of n.

l. What is the number of compositions of 24 into any number of parts, so that each part is divisible by three?

m. A function f is called monotone increasing if $x < y$ implies $f(x) \leq f(y)$. What is the number of monotone increasing functions from $[n]$ to $[n]$?

<u>For Feb. 5:</u> (remainder of the twelvefold way)

17. Write out all the partitions of $[4]$, and use this data to compute $\left\{ {4 \atop b} \right\}$ for all $b \in \mathbb{N}_0$, as well as $B(4)$.

18. A partition of an integer is similar to a composition of that integer, except order doesn't matter (i.e., $2+3$ and $3+2$ are the same partitions of 5). Write out all the partitions of 6, and use this data to compute $p(6)$, and also $p(6, b)$ for all $b \in \mathbb{N}_0$.

19. Find simple formulas for: (i) $\left\{ {a \atop a} \right\}$ (for $a \geq 0$); (ii) $\left\{ {a \atop 1} \right\}$ (for $a \geq 1$); and (iii) $\left\{ {a \atop 2} \right\}$ (for $a \geq 2$).

20. Prove the six formulas in the bottom two rows of the twelvefold way.

n. Find a simple formula for $\left\{ {a \atop 3} \right\}$ (for $a \geq 3$).

o. Find a simple formula for $\left\{ {a \atop a-2} \right\}$ (for $a \geq 2$).

p. Find a simple formula for $p(a, 2)$ (for $a \geq 2$).

<u>For Feb. 10:</u> (binomial coefficients)

21. Suppose that $a, b \in \mathbb{Z}$. Prove both of the following identities:
(i) If $a \geq 0$, then $\binom{a}{b} = \binom{a}{a-b}$; and (ii) If $a \geq b \geq 0$, then $\binom{a}{b} = \frac{a!}{b!(a-b)!}$.

22. Suppose that $b \in \mathbb{Z}$. Prove that $\binom{a}{b} = \binom{a-1}{b} + \binom{a-1}{b-1}$.
Warning: Do not assume that $a \in \mathbb{Z}$!

23. Let $b \in \mathbb{Z}$. Prove that $\binom{a}{b} = (-1)^b \binom{b-a-1}{b}$.

24. Let $a \in \mathbb{Z}$. Express $\binom{a}{|a|+1}$ without falling/rising powers (you may use factorials).

q. Suppose that b is a nonzero integer. Prove that $\binom{a}{b} = \frac{a}{b}\binom{a-1}{b-1}$.

r. Let $a \in \mathbb{Z}$. Express $\binom{a}{|a|}$ without falling/rising powers (you may use factorials).

s. Let $a, b \in \mathbb{N}_0$. Prove that $\sum_{i=0}^{a} \binom{i}{b} = \binom{a+1}{b+1}$. (Hint: induction on a)

t. Let $a \in \mathbb{R}$ and $b \in \mathbb{N}_0$. Prove that $\sum_{i=0}^{b} \binom{a+i}{i} = \binom{a+b+1}{b}$.
(Hint: induction on b)

For Feb. 12: (Stirling numbers of 2nd kind and Bell numbers)

25. Let $a, b \in \mathbb{N}_0$. Prove that $b^a = \sum_{i=0}^{a} \left\{{a \atop i}\right\} b^{\underline{i}}$, by counting functions from $[a]$ to $[b]$ in two different ways.
NOTE: This means that the two polynomials x^a and $\sum_{i=0}^{a} \left\{{a \atop i}\right\} x^{\underline{i}}$ agree on infinitely many values of x; hence they are equal for all x.

26. Suppose that $a > b > 0$. Prove that $\left\{{a+1 \atop b}\right\} = b\left\{{a \atop b}\right\} + \left\{{a \atop b-1}\right\}$.

27. Let $a \in \mathbb{N}_0$. Prove that $x^a = \sum_{i=0}^{a} \left\{{a \atop i}\right\} x^{\underline{i}}$, by induction and problem 26.

28. Prove that $B(a+1) = \sum_{i=0}^{a} \binom{a}{i} B(i)$, for all $a \in \mathbb{N}_0$.

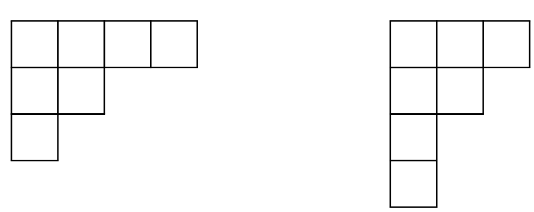
u. Prove that $B(a) < a!$ for all $a \geq 3$.

v. Prove that $b! < \left\{{2b \atop b}\right\} < (2b)!$ for all $b \geq 2$.

w. Use problem 26 to calculate the next column of the table of Stirling numbers of the 2nd kind in the back (i.e., for $a = 12$).

For Feb. 17: (integer partitions)

Given a partition of integer n, we may sort the values in decreasing order (i.e., $x_1 \geq x_2 \geq \cdots \geq x_k \geq 1$, with $n = x_1 + x_2 + \cdots + x_k$, then build a shape with x_i squares on row x_i. This shape is called a Ferrers diagram. The conjugate of a Ferrers diagram is obtained by reflecting it on its main diagonal. This too must be a partition. For example, $7 = 4+2+1$ and $7 = 3+2+1+1$ have conjugate Ferrers diagrams:



29. Draw the Ferrers diagrams for each partition of 6. Pair up each with its conjugate, and identify the self-conjugate ones. Then, find just the self-conjugate partitions of $7, 8, 9$.

30. Let $a, b \in \mathbb{N}$. Prove that $p(a, b)$, the number of partitions of a into exactly b parts, is equal to the number of partitions of a into parts, the largest of which is size b.

31. Let $a \in \mathbb{N}$. Prove that the number of self-conjugate partitions of a is equal to the number of partitions of a into distinct odd parts.

32. Let $a \geq 2$. Prove that the number of partitions of a in which each part is at least two, is exactly $p(a) - p(a-1)$.

x. Let $a, b \in \mathbb{N}$. Prove that the number of partitions of a into at most b parts, is equal to the number of partitions of a into parts, each no larger than b.

y. Let $a, b \in \mathbb{N}$. Prove that the number of partitions of a into at most b parts, is equal to the number of partitions of $a + b$ into exactly b parts.

z. Let $a \geq 4$. Find the number of partitions of a in which the difference of the first two parts is at least three.

Unit 2: Generating Functions and Recurrences

Ordinary generating functions: $A(x) = \sum_{i\geq 0} a_i x^i$

$(1+x)^a = \sum_{i\geq 0} \binom{a}{i} x^i$ $(a \in \mathbb{R})$
(binomial theorem)

$\frac{1}{1-x} = \sum_{i\geq 0} x^i = \sum_{i\geq 0} 1x^i$

$\frac{1}{(1-x)^{a+1}} = \sum_{i\geq 0} \binom{a+i}{a} x^i$ $(a \in \mathbb{N}_0)$

$\frac{1}{(1-rx)^{a+1}} = \sum_{i\geq 0} \binom{a+i}{a} r^i x^i$ $(a \in \mathbb{N}_0)$

$\frac{x^a}{(1-x)^{a+1}} = \sum_{i\geq 0} \binom{i}{a} x^i$ $(a \in \mathbb{N}_0)$

$e^x = \sum_{i\geq 0} \frac{1}{i!} x^i$

$-\ln(1-x) = \sum_{i\geq 1} \frac{1}{i} x^i$

Sum: $A(x) + B(x) = \sum_{i\geq 0} (a_i + b_i) x^i$

Product: $A(x)B(x) = \sum_{i\geq 0} \left(\sum_{j=0}^{i} a_j b_{i-j}\right) x^i$ (split $[i]$ into two intervals, possibly empty. Do a on the first subinterval and b on the second)

Shift: $x^k A(x) = \sum_{i\geq 0} a_i x^{k+i} = \sum_{i\geq k} a_{i-k} x^i$ $(k \in \mathbb{Z})$

Derivative: $A'(x) = \sum_{i\geq 0} i a_i x^{i-1} = \sum_{i\geq 1} i a_i x^{i-1} = \sum_{i\geq 0} (i+1) a_{i+1} x^i$

Antiderivative: $\int A(x)dx = C + \sum_{i\geq 0} \frac{a_i}{i+1} x^{i+1} = C + \sum_{i\geq 1} \frac{a_{i-1}}{i} x^i$

OGF Composition Thm: Suppose that a_i counts some operation on a set of size i, $a_0 = 0$, $A(x) = \sum_{i\geq 0} a_i x^i$. Let b_j count the number of ways to split $[j]$ into any number of disjoint nonempty subintervals, then do a on each one. $b_0 = 1$, $B(x) = \sum_{j\geq 0} b_j x^j$. Then $B(x) = \frac{1}{1-A(x)}$.

Exponential generating functions: $A(x) = \sum_{i\geq 0} a_i \frac{x^i}{i!}$

Product: $A(x)B(x) = \sum_{i\geq 0} \left(\sum_{j=0}^{i} \binom{i}{j} a_i b_{i-j}\right) x^i$ (split $[i]$ into two disjoint subsets, possibly empty. Do a on the first subset and b on the second)

EGF Composition Thm: Suppose that a_i counts some operation on a set of size i, $a_0 = 0$, $A(x) = \sum_{i\geq 0} a_i \frac{x^i}{i!}$. Let b_j count the number of ways to split $[j]$ into a number of disjoint nonempty subsets, then do a on each one. $b_0 = 1$, $B(x) = \sum_{i\geq 0} b_i \frac{x^i}{i!}$. Then $B(x) = e^{A(x)}$.

Unit 2 Exercises

For Feb. 24 (To OGF's and back again)

1. Suppose $A(x) = \frac{1+x}{(1-2x)(1+3x)}$ is an OGF. Find the closed form for a_i. (Hint: partial fractions).

2. Suppose $A(x) = \frac{1+x}{(1-2x)^2}$ is an OGF. Find the closed form for a_i.

3. Let $a_i = i^2$. Find the OGF $A(x) = \sum_{i \geq 0} a_i x^i$. (Hint: start with derivative).

4. Let $a_i = \frac{i-1}{i!}$. Find the OGF $A(x) = \sum_{i \geq 0} a_i x^i$.

a. Suppose $A(x) = \frac{1+x}{1+x^2}$ is an OGF. Find the closed form for a_i.

b. Set $a_i = i3^i$. Find the OGF $A(x) = \sum_{i \geq 0} a_i x^i$.

c. Prove that $A(x) = -\ln(1-x)$ is the OGF for $a_i = \frac{1}{i}$ $(a_0 = 0)$.

d. Fix $k \in \mathbb{N}$, and set $a_i = i^{\overline{k}}$. Find the OGF $A(x) = \sum_{i \geq 0} a_i x^i$.

For Feb. 26 (recurrences)

1. Multiply recurrence relation through by x^i, sum over all i where the relation holds.
2. Rewrite in terms of GF. 3. Solve equation for GF.
4. Convert GF to closed form.

5. Let a_i satisfy $a_0 = 5, a_i = 4a_{i-1} - 10$ $(i \geq 1)$. Find the OGF.

6. Let F_i satisfy $F_0 = 0, F_1 = 1, F_i = F_{i-1} + F_{i-2}$ $(i \geq 2)$. Find the OGF.

7. Let a_i satisfy $a_0 = 2, a_1 = 1, a_i = a_{i-1} + 30a_{i-2}$ $(i \geq 2)$. Find the OGF.

8. Let a_i satisfy $a_0 = 1, a_1 = 12, a_i = 6a_{i-1} - 9a_{i-2}$ $(i \geq 2)$. Find the OGF.

e. Find closed forms for the recurrences in problems 5-8.

f. Let a_i satisfy $a_0 = 1, a_i = a_{i-1} + i(i-1)$ $(i \geq 1)$. Find the OGF.

g. Let a_i satisfy $a_0 = a_1 = 1$ and $a_i = \sum_{j=1}^{i} 2^j a_{i-j}$ $(i \geq 2)$. Find the OGF, and then the closed form for the recurrence (you will need cases to handle $i = 0, 1$).

For Mar. 3 (recurrences, continued)

9. Your habit in climbing stairs is to go up either one or two stairs at a time, at each step. Let $S(n)$ denote the number of ways to reach the n-th stair. Find a closed form for $S(n)$.

10. You deposit \$1000 into a savings account that pays two percent interest at the end of each year. At the beginning of each year, you deposit another \$500 into this shitty account. How much will be in this account after n years?

11. Five people start spreading a hilarious meme. At the start of each day, each person who has seen the meme sends it to two new people (who haven't yet). At the end of each day, one person who has seen the meme decides it is played out and doesn't send it anymore. How many people will be spreading the meme after n days?

12. "Tower of Hanoi" There are three pegs, and n disks of different sizes. The disks all start on one peg in order of size and we must move them to another. We move one disk at a time, and may never put a larger disk onto a smaller one. How many moves does it take?

h. Find the number of ways to color the squares of a $1 \times n$ chessboard using the colors red, white, and blue, so that no two red squares are adjacent.

i. Find the number of ways to color the squares of a $1 \times n$ chessboard using the colors red, white, and blue, so that no red square is adjacent to a white square.

j. Find the number of n-digit nonnegative integers in which no three consecutive digits are the same.

k. Find the OGF whose terms represent the number of ways to color the squares of a $1 \times n$ chessboard using the colors red, white, and blue, so that the specific sequence red-white-blue does not occur. You need not find a closed form for the OGF.

For Mar. 5 (binomial theorem)

13. Let $a \in \mathbb{N}_0$. Prove that $\sum_{i\geq 0} \binom{a}{2i} = \sum_{i\geq 0} \binom{a}{2i+1}$. (both are finite sums)

14. Let $a \in \mathbb{N}_0$. Prove that $\sum_{i=0}^{n} i\binom{a}{i} = a2^{a-1}$.

15. Prove the Chu-Vandermonde identity: Let $i \in \mathbb{N}_0$, $a, b \in \mathbb{R}$. Then $\binom{a+b}{i} = \sum_{j=0}^{i} \binom{a}{j}\binom{b}{i-j}$.

16. Let $b \in \mathbb{N}$. Simplify $\binom{1/2}{b}$ to ordinary powers and factorials. Use this to write down the Maclaurin series for $\sqrt{1+x}$.

l. Let $a \in \mathbb{N}_0$. Prove that $3^a = \sum_{i\geq 0} 2^i \binom{a}{i}$.

m. Use Maclaurin (Taylor) series to prove the binomial theorem.

n. Determine, with proof, which $a, x \in \mathbb{R}$ cause the series in the binomial theorem to converge.

For Mar. 10 (integer compositions)

17. We have six magenta balls, five plum balls, and six violet balls. Balls are identical except for color. Let a_n denote the number of ways of selecting n balls. Find the generating function, and use it to find a_4 and a_{10}.

18. We have six magenta balls, five plum balls, and six violet balls. Balls are identical except for color. Let a_n denote the number of ways of selecting n balls, with at least one of each color required. Find the generating function, and use it to find a_4 and a_{10}.

19. Let a_n denote the number of ways to make \$n using at least one two dollar bill, two or three five dollar bills, and any number of one dollar bills. Find the generating function, and use it to find a_{20}.

20. We want to count nonnegative integer solutions to $x_1 + x_2 + x_3 + x_4 = n$, where x_1 is even, $x_2 \geq 2$, x_3 is both even and at least 2, and $x_4 \leq 2$. Find a generating function, and use it to solve the problem for $n = 10$. (a computer may be helpful)

o. We have an unlimited supply of balls, of five different colors. Balls are identical except for color. Let a_n denote the number of ways of selecting n balls, with 1-3 of each color. Find the generating function, and use it to find a_{10}.

p. The *multinomial theorem* says that, for $a \in \mathbb{N}_0$,

$$(x_1 + x_2 + \cdots + x_m)^a = \sum \frac{a!}{k_1!k_2!\cdots k_m!} x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m},$$

where the sum is over all nonnegative integers $k_1, k_2, \ldots, k_m$ satisfying $k_1+k_2+\cdots+k_m = a$. Use it to prove the binomial theorem (for $a \in \mathbb{N}_0$).

q. Use the multinomial theorem instead of generating functions to solve problem o. (i.e. find a_{10}) again.

r. Use the multinomial theorem to count the number of distinct rearrangements of MISSISSIPPI.

For Mar. 12 (EGFs)

21. Consider the sequence $a_0 = 0$, $a_n = 2na_{n-1} + n!$ $(n \geq 1)$. Find the EGF $A(x)$, and use it to find a closed form for a_n.

22. Let a_n denote the number of ways of partitioning n people into groups of size 3 or 4. Find the EGF and use it to compute a_{17}. Groups are identical except for size; people are, obviously different.

23. Let a_n denote the number of ways of partitioning n people into Magenta Group, Plum Group, and Violet Group. We insist that Magenta Group have an even number of people (possibly zero), and Plum Group have an odd number of people. Find the EGF, and use it to compute a_{17}.

24. Let a_n denote the number of ways of partitioning n people into Magenta Group, Plum Group, and Violet Group (groups may not be empty, and people are, obviously, different). Find the EGF, and use it to compute a_{17}.

s. Let a_n denote the number of ways of partitioning n people into Magenta Group, Plum Group, and Violet Group, where each group must have an odd number of members. Find the EGF, and use it to compute a_{17}.

t. Consider the sequence $a_0 = 5$, $a_n = na_{n-1} + 2n$ $(n \geq 1)$. Find the EGF $A(x)$, and use it to find a closed form for a_n.

u. Recall the recurrence on Bell numbers $B(a+1) = \sum_{i=0}^{a} \binom{a}{i} B(i)$ $(a \in \mathbb{N}_0)$ that you proved in Exercise 1.28. Let $B(x) = \sum_{i \geq 0} B(i) \frac{x^i}{i!}$ denote the EGF for Bell numbers. Prove that this satisfies $B'(x) = e^x B(x)$, then solve this differential equation (using calculus and $B(0) = 1$) to get $B(x) = e^{e^x - 1}$.

<u>For Mar. 17</u> (OGF/EGF compositions)

25. Let a_n denote the number of compositions of the integer n into parts of size 2 or 3. Find the OGF and use it to compute a_{10}.

26. Let a_n denote the number of compositions of the integer n into odd parts. Find the OGF and use it to compute a_{10}.

27. The professor makes a list of n students, in alphabetical order. Then, this list is split at various places (perhaps none), resulting in nonempty sublists. Then the professor chooses one person from each sublist to receive extra credit. Let a_n denote the number of ways to do this. Find an OGF for a_n.

28. Let a_n denote the number of ways of partitioning n people into groups of size 3, 4 or 5. Find the EGF and use it to compute a_{10}.

v. Let a_n denote the number of compositions of the integer n into parts of size 1, 3, or multiples of 4. Find the OGF.

w. Let a_n denote the number of ways to partition a set of n people, and have each block sit around a circular table. (two seating arrangements are identical if each person has the same left neighbor in both). Find the EGF, and then a closed form for a_n.

x. The professor takes n students, and divides this into nonempty subsets. Then the professor chooses one person from each subset to receive extra credit. Let a_n denote the number of ways to do this. Find an EGF for a_n.

Unit 3: Finite Calculus

Difference (discrete derivative): $\Delta F(n) = F(n+1) - F(n)$
Indefinite antidifference: $\Delta F(n) = f(n)$ iff $F(n) = C + \sum f(n)$
Definite antidiff.: $\sum_a^b f(n)\delta n = \sum\limits_{a \le n < b} f(n) = \sum_{n=a}^{b-1} f(n)$ $(a, b \in \mathbb{Z},\ a \le b)$
Fundamental theorem of difference calculus (FTDC): $(a, b \in \mathbb{Z}$ with $a \le b)$
$\sum_a^b f(n)\delta n = F(n)|_a^b = F(b) - F(a)$, where $F(n)$ satisfies $\Delta F(n) = f(n)$.

$\Delta(n^{\underline{a}}) = an^{\underline{a-1}}$ $(a \in \mathbb{Z})$, $\quad n^{\underline{a+b}} = n^{\underline{a}}(n-a)^{\underline{b}}$ $(a, b \in \mathbb{Z})$
$n^a = \sum_{i=0}^a \left\{ {a \atop i} \right\} n^{\underline{i}}$ $(a \in \mathbb{N}_0)$, $\quad \Delta(H_n) = n^{\underline{-1}}$, $\quad \Delta(c^n) = (c-1)c^n$ $(c \in \mathbb{R})$
$\Delta(cF(n)) = c\Delta(F(n))$ and $\Delta(F(n) + G(n)) = \Delta(F(n)) + \Delta(G(n))$

product rule:
$\Delta(F(n)G(n)) = F(n)\Delta(G(n)) + G(n+1)\Delta(F(n))$

indefinite summation by parts:
$\sum F(n)\Delta G(n) = F(n)G(n) - \sum G(n+1)\Delta F(n)$

definite summation by parts:
$\sum_a^b F(n)\Delta G(n)\delta n = F(n)G(n)|_a^b - \sum_a^b G(n+1)\Delta F(n)$

Gosper: Install sagemath locally or visit `https://sagecell.sagemath.org/`
This first part of this code will produce an antidifference to $f(n)$ using Gosper's algorithm. The second part (optional, the last two lines) will compute a sum, using sagemath's many algorithms (including Gosper's).

```
a,n,Delta = var('a,n,Delta')
f = ⟨insert f(n) here⟩
show(f,"=",Delta,factor(f*f.gosper_term(n)))
sum_verify = sum(f,n,0,a,hold=True)
show("\n\n\n\n\n\n",sum_verify==factor(sum_verify.unhold()))
```

Unit 3 Exercises

For Mar. 24 (powers and exponents)

1. Prove that $\Delta(c^n) = (c-1)c^n$, for all $c \in \mathbb{R}$, and that $\Delta(H_n) = n^{\underline{-1}}$.

2. Prove that $\Delta(n^{\underline{a}}) = an^{\underline{a-1}}$ for all $a \in \mathbb{Z}$) (you will need different proofs for positive and negative a).

3. Prove that $n^{\underline{a+b}} = n^{\underline{a}}(n-a)^{\underline{b}}$ for all $a, b \in \mathbb{Z}$. You will have four different cases for a, b positive/negative.

4. Compute, with proof $\Delta(n^{\overline{a}})$, for $a \in \mathbb{N}_0$.

a. How should we define $n^{\overline{-a}}$, for $a \in \mathbb{N}_0$, to maintain the formula you found in problem 4?

b. Find an analog of $n^{\underline{a+b}} = n^{\underline{a}}(n-a)^{\underline{b}}$, but for rising factorials instead of falling factorials, that holds for all $a, b \in \mathbb{Z}$. Prove your result.

c. Find antidifferences for $c^n, n^{\underline{a}}, n^{\overline{a}}$. (Careful, the answer depends on a.)

For Mar. 26 (fundamental theorem)

5. Prove the FTDC. (induction)

6. Let $a, b, c \in \mathbb{Z}$. (i) Suppose that $a \leq b \leq c$. Prove that $\sum_a^b f(n)\delta n + \sum_b^c f(n)\delta n = \sum_a^c f(n)\delta n$. (ii) Suppose we want the identity from (i) to be true for all $a, b, c \in \mathbb{Z}$. How should we define $\sum_a^b f(n)\delta n$ if $a > b$? (iii) What does the FTDC say if $a > b$?

7. Let $a, b, k \in \mathbb{Z}$. Use the FTDC to find $\sum_a^b n^{\underline{k}}\delta n$.

8. Let $a \in \mathbb{N}$. Use the FTDC to find $\sum_{n=1}^a n^2$.
Hint: first express n^2 in terms of falling powers.

d. Let $a \in \mathbb{N}$. Use the FTDC to find $\sum_{n=1}^a n^3$.

e. Let $a \in \mathbb{N}$. Use the FTDC to find $\sum_{n=2}^a \frac{1}{(n-1)n(n+1)}$.

f. Let $a \in \mathbb{Z}$ with $a \geq 2$. Use partial fractions to find $\sum_{n=2}^a \frac{1}{(n-1)n(n+1)}$.

For Apr. 7 (infinite sums)
We can calculate $\sum_a^\infty f(n)\delta n$ as $\lim\limits_{N\to\infty} \sum_a^N f(n)\delta n$.

9. Let $c \in \mathbb{R}$. Find $\sum_{n\geq 0} c^n = \sum_0^\infty c^n \delta n$.

10. Find $\sum_{n\geq 0} \frac{1}{(n+1)(n+2)}$.

11. Find $\sum_{n\geq 2} \frac{1}{(n-1)n(n+1)}$.

12. Prove that $\Delta \frac{n+1}{2^{n-1}} = -\frac{n}{2^n}$, and use this to find $\sum_{n\geq 0} \frac{n}{2^n}$.

g. Prove that $\Delta\frac{2n^2-1}{2(n^2-n)} = -\frac{1}{(n+1)(n-1)}$, and use this to find $\sum_{n\geq 2}\frac{1}{(n+1)(n-1)}$.

h. Prove that $\Delta\frac{2n+1}{4\cdot 3^{n-1}} = -\frac{n}{3^n}$, and use this to find $\sum_{n\geq 0}\frac{n}{3^n}$.

i. Prove that $\Delta\frac{n^2+n+2}{2^{n-1}} = -\frac{n^{\underline{2}}}{2^n}$, and use this to find $\sum_{n\geq 0}\frac{n^{\underline{2}}}{2^n}$.

j. Prove that $\Delta\frac{n^2+2n+3}{2^{n-1}} = -\frac{n^2}{2^n}$, and use this to find $\sum_{n\geq 0}\frac{n^2}{2^n}$.

<u>For Apr. 9</u> (product rule and summation by parts)

13. Use the product rule to compute $\Delta(2^n n^{\underline{k}})$ and $\Delta 2^n H_n$.

14. Repeat problem 13, reversing the order of the multiplicands. Prove that you get the same answer.

15. Use summation by parts on $\sum n2^n\delta n$, then evaluate that simpler sum to find the general antidifference for $n2^n$.

16. Use summation by parts on $\sum nH_n\delta n$, then evaluate that simpler sum to find the general antidifference for nH_n.

k. Find $\sum_0^a n\cdot n^{\underline{3}}\,\delta n$.

l. Find $\sum_1^a H_n n^{\underline{3}}\,\delta n$.

m. Find $\sum_0^a n^{\underline{2}}3^n\delta n$. (Hint: twice)

<u>For Apr. 14</u> (Gosper's algorithm)
Gosper's algorithm finds an antidifference for many hypergeometric functions. We will not study its details here. We will use it as a magic black box. If you'd like to learn more, as well as other even more amazing tools, read the free book A=B.

17. Use Gosper's algorithm to find an antidifference for $f(n) = n\cdot n!$. Then verify it by hand (taking its difference to get $f(n)$ back).

18. Use Gosper's algorithm to rediscover the antidifferences you were given in exercises 12,g,h,i,j. Then go back and solve exercises 10,11 using Gosper's algorithm.

19. Use Gosper's algorithm to create three new, convergent, infinite sums of the above type, and evaluate each.

20. Find $\sum\frac{n^2 4^n}{(n+2)(n+1)}\delta n$.

n. Use Gosper's algorithm to find $\sum_{n=1}^{\infty}\frac{n-1}{n(n+1)(n+2)}\delta n$.

o. Find $\sum\frac{n^4 4^n}{\binom{2n}{n}}\delta n$.

p. Find $\sum\frac{(4n-1)\binom{2n}{n}^2}{(2n-1)^2 4^{2n}}\delta n$.

Unit 4: Burnside's Lemma

We fix $n \in \mathbb{N}$. A *permutation* of $[n]$ is a bijection $\sigma : [n] \to [n]$.
We can write a permutation in two ways:
Two-line notation. Enclosed in parentheses, the domain goes on the first line (in order), and the image of σ on the second line.
Cycle notation. We put each cycle, in order, enclosed in parentheses. Cycles of length one, aka fixed points, i.e. x such that $\sigma(x) = x$, are omitted. The numbers within eacy cycle may be permutated cyclically, and the cycles themselves may be written in any order. *Standard form* orders each cycle to start with the smallest element in that cycle, and orders the cycles in increasing order of first element.

Example: $n = 6$, $\sigma = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 1 & 4 & 3 & 2 \end{smallmatrix}\right) = (2\ 6)(5\ 3\ 1) = (1\ 5\ 3)(2\ 6)$

Composition. Two permutations may be composed (as functions) giving another permutation[3]. Example: $\sigma \circ \sigma = (1\ 3\ 5)$

Groups. The set of all permutations of $[n]$, together with the composition operation, forms a *group*, called S_n. Key properties:
(i) There is an identity id satisfying $id \circ \sigma = \sigma \circ id = \sigma$ (for all $\sigma \in S_n$);
(ii) Every $\sigma \in S_n$ has an inverse $\sigma^{-1} \in S_n$ satisfying $\sigma \circ \sigma^{-1} = \sigma^{-1} \circ \sigma = id$.

Permutation Groups. Take G to be some subset of S_n, that is closed under $\circ$ and has both "key properties" above. We call G a *permutation group*.
Example: $G = \{(1\ 2), (1\ 2) \circ (1\ 2) = id\}$ has $|G| = 2$.

Colorings of $[n]$. We fix a set of colors C, and set X to be the set of all colorings of each element of $[n]$.
Example: $C = \{red, blue\}$, $X = \{[1 = 2 = 3 = 4 = 6 = red, 5 = blue], \ldots\}$.

Fixed Sets. Given set of colorings X and $\sigma \in S_n$, set $\boldsymbol{Fix(\sigma)}$ to be the set of elements of X that are fixed by σ.

Orbits. Given G, X, an *orbit* is a minimal subset of X, permuted by G. Example: $n = 3, C = \{red, blue\}, G = \{(1\ 2), id\}$, one orbit is $\{[1 = red, 2 = blue, 3 = red], [1 = blue, 2 = red, 3 = red]\}$

Burnside's Lemma. Given G, X, the # of orbits in X is $\frac{1}{|G|}\sum_{\sigma \in G} |Fix(\sigma)|$.

[3]WARNING: $\sigma \circ \sigma' \neq \sigma' \circ \sigma$ in general.

Unit 4 Exercises

For Apr. 21 (permutations)
All of these problems will be in S_6, and use $\sigma = (1\ 5\ 3)(2\ 6)$, as in the example on the previous page.

1. Take $\tau = (1\ 2)$. Compute $\sigma \circ \tau$ and $\tau \circ \sigma$. Write each in both two-line notation and in standard-form cycle notation.

2. Compute $\sigma, \sigma^2 = \sigma \circ \sigma, \sigma^3 = (\sigma^2) \circ \sigma, \sigma^4 = (\sigma^3) \circ \sigma, \ldots$, until it repeats. We call this set $\langle \sigma \rangle$, and it is a permutation group. Write each element of $\langle \sigma \rangle$ in both two-line notation and in standard-form cycle notation.

3. Take $\tau = (2\ 6)$. Compute $\langle \sigma, \tau \rangle$, the smallest permutation group containing both σ and τ (be sure it is closed under $\circ$). Starting now, unless specified otherwise, always give permutations given in standard-form cycle notation.

4. For $\tau = (1\ 2)$, compute each of: σ^{-1}, τ^{-1}, $(\sigma \circ \tau)^{-1}$, $(\tau \circ \sigma)^{-1}$, $\sigma^{-1} \circ \tau^{-1}$, and $\tau^{-1} \circ \sigma^{-1}$. Can you prove the pattern you observe?

a. Compute $|\langle \sigma \rangle|$ from problem 2, $|\langle \sigma, \tau \rangle|$ from problem 3, and $|S_6|$ for all of these.

b. Let $T_1 = \{(a\ b) : a, b \in [n]\}$, the set of all "transpositions" (permutations which swap two numbers and fix everything else). Prove that $\langle T_1 \rangle = S_n$.

c. Let $T_2 = \{(1\ 2), (2\ 3), \ldots, (n-1\ n)\}$, a set of $n-1$ transpositions. Prove that $\langle T_2 \rangle = S_n$.

d. Let $T_3 = \{(1\ 2), (1\ 2\ 3\ \cdots\ n)\}$, a set with just two permutations. Prove that $\langle T_3 \rangle = S_n$.

For Apr. 23 (Stirling numbers of the first kind)

5. Write down each element of S_4, group by how many cycles they contain, and use this data to compute $\left[{4 \atop b}\right]$ for all $b \in \mathbb{N}_0$.

6. Let $a \in \mathbb{N}$. Prove that $\left[{a \atop 1}\right] = (n-1)!$, $\left[{a \atop a}\right] = 1$, and $\left[{a \atop a-1}\right] = \binom{a}{2}$.

7. Prove that Stirling numbers of the first kind satisfy $\left[{a \atop b}\right] = (a-1)\left[{a-1 \atop b}\right] + \left[{a-1 \atop b-1}\right]$, for all $a, b \in \mathbb{N}$. (Hint: Fix b, induction on a)

8. Let $a \in \mathbb{N}$. Prove that $\left[{a \atop 2}\right] = (a-1)! H_{a-1}$.

e. Let $a, b \in \mathbb{N}_0$. Prove that $\left[{a \atop b}\right] \geq \left\{{a \atop b}\right\}$.

f. Let $a \in \mathbb{N}_0$. Prove that $n^{\overline{a}} = \sum_{i=0}^{a} \left[{a \atop i}\right] n^i$.

g. Let $a \in \mathbb{N}_0$. Prove that $n^{\underline{a}} = \sum_{i=0}^{a} \left[{a \atop i}\right] (-1)^{a-i} n^i$.

For Apr. 28 (colorings and fixed sets)

9. Let $n = 3$ and $C = \{red, blue\}$. Write down all of X (the set of all colorings of $[3]$). Determine which of them are in $Fix(\tau)$, for every $\tau \in \langle \sigma \rangle$, where $\sigma = (1\ 2\ 3)$.

10. Let $n = 3$ and $C = \{red, blue\}$. Again write down all of X. Determine which of them are in $Fix(\tau)$, for every $\tau \in S_3 \setminus \langle\sigma\rangle$, where $\sigma = (1\ 2\ 3)$.

11. Let $n = 3$ but now C is unknown, with $|C| = m$. Determine $|Fix(\tau)|$, for every $\tau \in \langle\sigma\rangle$, where $\sigma = (1\ 2\ 3)$.

12. Let $n = 3$, and C is still unknown with $|C| = m$. Determine $|Fix(\tau)|$, for every $\tau \in S_3 \setminus \langle\sigma\rangle$, where $\sigma = (1\ 2\ 3)$.

h. Let $n \geq 3$ and C is unknown with $|C| = m$. Determine $|Fix(\tau)|$, for every $\tau \in \langle\sigma\rangle$, where $\sigma = (1\ 2\ 3)$.

i. Let $n \geq 4$ and C is unknown with $|C| = m$. Set $\sigma = (1\ 2\ 3\ 4)$, and determine $|Fix(\tau)|$, for every $\tau \in \langle\sigma\rangle$. There will be a surprise!

j. Let $n \geq 5$ and C is unknown with $|C| = m$. Set $\sigma = (1\ 2\ 3\ 4\ 5)$, and determine $|Fix(\tau)|$, for every $\tau \in \langle\sigma\rangle$.

For Apr. 30 (Burnside's Lemma)

13. We color the edges of an equilateral triangle from $C = \{red, white, blue\}$. Determine how many different colorings there are, up to rotation. Draw them all.

14. We build a (short) necklace with three beads, each from $C = \{red, white, blue\}$. Determine how many different necklaces there are, up to rotation and reflection. Draw them all.

15. We color the edges of a square, chosen from m colors. Determine how many squares there are, up to rotation.

16. We color the edges and corners of a square, chosen from m colors. Determine how many squares there are, up to rotation.

k. We color the edges of a square, chosen from m colors. Determine how many squares there are, up to rotation and reflection.

l. We color the edges and corners of a square, chosen from m colors. Determine how many squares there are, up to rotation and reflection.

m. We color the edges of a pentagon, chosen from m colors. Determine how many pentagons there are, up to rotation.

n. Let p be an odd prime. We color the edges and vertices of a p-gon, chosen from m colors. Determine how many there there are, up to rotation.

For May 5 (Burnside's Lemma, continued)

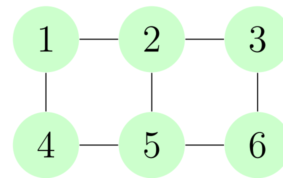
17. We color the faces of a tetrahedron from $C = \{red, white, blue\}$. Determine how many different colorings there are, up to rotation.

18. We color the edges of a tetrahedron from $C = \{red, white, blue\}$. Determine how many different colorings there are, up to rotation.

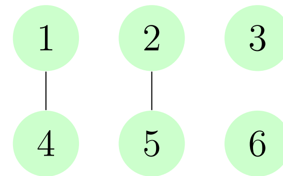
19. We color the vertices of a tetrahedron from $C = \{red, white, blue\}$. Determine how many different colorings there are, up to rotation.

20. We color the faces of a cube from $C = \{red, white, blue\}$. Determine how many different colorings there are, up to rotation.

o. Find all permutations of this graph. How many colorings are there with $C = \{red, white\}$?



p. Find all permutations of this graph. How many colorings are there with $C = \{red, white\}$?



q. Color each face of a dodecahedron (regular 12-sided solid) from $C = \{red, white\}$. How many colorings there are, up to rotation.

Notation

Set is a collection of distinct elements, order doesn't matter. $\{a, b, c\} = \{b, a, a, c\}$

Multiset is a collection of elements, repeats allowed, order doesn't matter. $\{a^3, b^2\} = \{b^2, a^3\}$

Cardinality of S is the number of elements it contains, counting multiplicity. S can be a set or multiset. $|\{a, b, c\}| = 3 = |\{a^2, b^1\}|$

List/Tuple/Word is a collection of elements, repeats allowed, order matters. $(a, b) \neq (a, b, a) \neq (a, a, b)$

List Without Repetition is a collection of distinct elements, order matters. $(a, b, c) \neq (a, c, b)$

Partition of set S is a set of nonempty sets (called parts) $\{P_1, \ldots, P_k\}$ satisfying (i) $P_1 \cup \cdots \cup P_k = S$; and (ii) $\forall i \neq j,\ P_i \cap P_j = \emptyset$.

Floor For each $x \in \mathbb{R}$, there is a unique integer $\lfloor x \rfloor$, the floor of x, satisfying $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$.

Ceiling For each $x \in \mathbb{R}$, there is a unique integer $\lceil x \rceil$, the ceiling of x, satisfying $\lceil x \rceil - 1 < x \leq \lceil x \rceil$.

$[\boldsymbol{n}] = \{1, 2, \ldots, n\}$, a set of n integers ($n \in \mathbb{N}_0$). $[0] = \emptyset, [1] = \{1\}$

$\boldsymbol{x^{\underline{k}}} = x(x-1)\cdots(x-k+1)$, x to the k falling ($k \in \mathbb{N}_0$). $x^{\underline{0}} = 1, x^{\underline{1}} = x$

$\boldsymbol{x^{\underline{-k}}} = \frac{1}{(x+1)(x+2)\cdots(x+k)}$, ($k \in \mathbb{N}_0$).

$\boldsymbol{x^{\overline{k}}} = x(x+1)\cdots(x+k-1)$, x to the k rising ($k \in \mathbb{N}_0$). $x^{\overline{0}} = 1, x^{\overline{1}} = x$

$\boldsymbol{n!} = n^{\underline{n}} = n(n-1)\cdots 1$, n factorial ($n \in \mathbb{N}_0$). $0! = 1$

$\binom{\boldsymbol{a}}{\boldsymbol{b}} = \frac{a^{\underline{b}}}{b!}$, binomial coefficients ($b \in \mathbb{Z}$). We define $\binom{a}{b} = 0$ for $b < 0$.

$\left(\!\binom{\boldsymbol{a}}{\boldsymbol{b}}\!\right) = \frac{a^{\overline{b}}}{b!}$, multinomial coefficients ($b \in \mathbb{Z}$). We define $\left(\!\binom{a}{b}\!\right) = 0$ for $b < 0$.

$\boldsymbol{\left\{ a \atop b \right\}}$ Stirling number of the second kind ($a, b \in \mathbb{N}_0$). # of partitions of $[a]$ into exactly b parts

$\boldsymbol{\left[a \atop b \right]}$ Stirling number of the first kind ($a, b \in \mathbb{N}_0$). # of permutations of $[a]$ with exactly b cycles

$\boldsymbol{B(a)}$ Bell number ($a \in \mathbb{N}_0$). # of partitions of $[a]$, $B(a) = \sum_{i=1}^{a} \left\{ a \atop i \right\}$

$\boldsymbol{p(a, b)}$ restricted partition function ($a, b \in \mathbb{N}_0$). # of partitions of a into exactly b parts.

$\boldsymbol{p(a)}$ partition function ($a \in \mathbb{N}_0$). # of partitions of a, $p(a) = \sum_{i=1}^{a} p(a, i)$

$\boldsymbol{H_a} = \sum_{i=1}^{a} \frac{1}{i}$, harmonic numbers ($a \in \mathbb{N}_0$). $H_0 = 0, H_1 = 1, H_2 = 1 + \frac{1}{2}$

$\boldsymbol{A(x) = \sum_{i \geq 0} a_i x^i}$, ordinary generating function for sequence $\{a_i\}$

$\boldsymbol{B(x) = \sum_{i \geq 0} b_i \frac{x^i}{i!}}$, exponential generating function for sequence $\{b_i\}$

$b\backslash a$	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2		1	3	6	10	15	21	28	36	45	55	66
3			1	4	10	20	35	56	84	120	165	220
4				1	5	15	35	70	126	210	330	495
5					1	6	21	56	126	252	462	792
6						1	7	28	84	210	462	924
7							1	8	36	120	330	792
8								1	9	45	165	495
9									1	10	55	220
10										1	11	66
11											1	12
12												1

Binomial coefficients.

$\binom{a}{b}$ appears in column a, row b

$\binom{a}{0} = 1$ (all a)

$b\backslash a$	2	3	4	5	6	7	8	9	10	11
2	1	3	7	15	31	63	127	255	511	1023
3		1	6	25	90	301	966	3025	9330	28501
4			1	10	65	350	1701	7770	34105	145750
5				1	15	140	1050	6951	42525	246730
6					1	21	266	2646	22827	179487
7						1	28	462	5880	63987
8							1	36	750	11880
9								1	45	1155
10									1	55
11										1

Stirling numbers of the 2nd kind.

$\left\{{a \atop b}\right\}$ appears in column a, row b

$\left\{{a \atop 0}\right\} = \begin{cases} 1 & \text{if } a=0 \\ 0 & \text{if } a\geq 1 \end{cases}$

$\left\{{a \atop 1}\right\} = \begin{cases} 0 & \text{if } a=0 \\ 1 & \text{if } a\geq 1 \end{cases}$

$b\backslash a$	1	2	3	4	5	6	7	8	9	10
1	1	1	2	6	24	120	720	5040	40320	362880
2		1	3	11	50	274	1764	13068	109584	1026576
3			1	6	35	225	1624	13132	118124	1172700
4				1	10	85	735	6769	67284	723680
5					1	15	175	1960	22449	269325
6						1	21	322	4536	63273
7							1	28	546	9450
8								1	36	870
9									1	45
10										1

Stirling numbers of the 1st kind.

$\left[{a \atop b}\right]$ appears in column a, row b

$\left[{a \atop 0}\right] = \begin{cases} 1 & \text{if } a=0 \\ 0 & \text{if } a\geq 1 \end{cases}$

All blank entries denote 0.