MATH 579 Exam 7 Solutions

1. How many integers in [123] are relatively prime to 10?

   Let $S = \{s_2, s_5\}$ where s_2 denotes "is a multiple of 2", and s_5 denotes "is a multiple of 5". We seek $f_=(\emptyset) = f_\geq(\emptyset) - f_\geq(\{s_2\}) - f_\geq(\{s_5\}) + f_\geq(\{s_2, s_5\}) = 123 - \lfloor \frac{123}{2} \rfloor - \lfloor \frac{123}{5} \rfloor + \lfloor \frac{123}{10} \rfloor = 123 - 61 - 24 + 12 = 50$.

2. (5-10 points) How many permutations of length n contain exactly two 1-cycles?

   There are $\binom{n}{2}$ ways to pick the 1-cycles, and the rest must contain no 1-cycles; there are D_{n-2} ways to do that. Hence the answer is $\frac{n(n-1)}{2}(n-2)!(1 - \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{(-1)^{n-2}}{(n-2)!}) = \frac{n!}{2}(1 - \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{(-1)^{n-2}}{(n-2)!})$.

   As a curiosity, note that this is approximately $\frac{1}{2e}$ of the total number of permutations, while around $\frac{1}{e}$ of them have no 1-cycles, and around $\frac{1}{e}$ of them have exactly one 1-cycle. Together, these three make up approximately $\frac{5}{2e} \approx 92\%$ of all permutations.

3. (5-10 points) How many 2×2 matrices are there with entries from the set $\{0, 1, 2, 3\}$ that contain no 0-rows and no 0-columns?

   Let $S = \{r_1, r_2, c_1, c_2\}$, where r_1 denotes "row 1 is a 0-row", etc. We seek $f_=(\emptyset)$. $f_\geq(\emptyset) = 4^4 = 256$. $f_\geq(\{r_1\}) = f_\geq(\{r_2\}) = f_\geq(\{c_1\}) = f_\geq(\{c_2\}) = 4^2 = 16$. $f_\geq(\{r_1, r_2\}) = f_\geq(\{c_1, c_2\}) = 1$, while $f_\geq(\{r_i, c_j\}) = 4$. If $|T| \geq 3$, then $f_\geq(T) = 1$. Putting it all together, $f_=(\emptyset) = 256 - 4 \cdot 16 + 2 \cdot 1 + 4 \cdot 4 - \binom{4}{3}1 + \binom{4}{4}1 = 207$.

4. (5-10 points) What is the number of integral solutions of the equation $x_1 + x_2 + x_3 = 15$ that satisfy $0 \leq x_1 \leq 5$, $0 \leq x_2 \leq 7$, $0 \leq x_3 \leq 10$?

   Let $S = \{s_1, s_2, s_3\}$, where s_1 denotes "$x_1 \geq 6$", s_2 denotes "$x_2 \geq 8$", s_3 denotes "$x_3 \geq 11$". We seek $f_=(\emptyset) = f_\geq(\emptyset) - f_\geq(\{s_1\}) - f_\geq(\{s_2\}) - f_\geq(\{s_3\}) + f_\geq(\{s_1, s_2\}) + f_\geq(\{s_1, s_3\}) + f_\geq(\{s_2, s_3\}) - f_\geq(\{s_1, s_2, s_3\}) = \left(\!\binom{3}{15}\!\right) - \left(\!\binom{3}{9}\!\right) - \left(\!\binom{3}{7}\!\right) - \left(\!\binom{3}{4}\!\right) + \left(\!\binom{3}{1}\!\right) = 33$.

5. (5-12 points) What is the number of integral solutions of the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 50$ that satisfy $5 \leq x_i \leq 15$ for $i = 1, 2, 3, 4, 5$?

   Set $y_i = x_i - 5$; we get $y_1 + y_2 + y_3 + y_4 + y_5 = 25$ and $0 \leq y_i \leq 10$. Set $S = \{s_1, s_2, s_3, s_4, s_5\}$, where s_i denotes "$y_i \geq 11$". Note that, for $T \subseteq S$, $f_=(T)$ depends only on $|T| = i$. Hence we set $a(5 - i) = f_=(T), b(5 - i) = f_\geq(T)$. We calculate $b(5) = \left(\!\binom{5}{25}\!\right) = 23751$, $b(4) = \left(\!\binom{5}{14}\!\right) = 3060$, $b(3) = \left(\!\binom{5}{3}\!\right) = 35$, and $b(2) = b(1) = b(0) = 0$. We seek $a(5) = \binom{5}{5}(-1)^{5-5}b(5) + \binom{5}{4}(-1)^{5-4}b(4) + \binom{5}{3}(-1)^{5-3}b(3) + 0 + 0 + 0 = 23751 - 5 \cdot 3060 + 10 \cdot 35 = 8801$.