MATH 579 Exam 4 Solutions

1. How many subsets of $[30]$ are larger than their complements?

   $\binom{30}{15}$ of the subsets have its complement of the same size, while the remaining ones, of which there are $2^{30} - \binom{30}{15}$, have its complement of a different size. Half of these are smaller, half are bigger, so the answer is $\frac{2^{30}-\binom{30}{15}}{2} = 459,312,152$.

2. Prove that $\sum_k (-4)^k \binom{100}{k} = 3^{100}$.

   Using the binomial theorem we get $\sum_k (-4)^k \binom{100}{k} 1^{100-k} = (-4+1)^{100} = (-3)^{100} = 3^{100}$.

3. How many northeastern lattice paths are there from $(0,0)$ to $(15,10)$ that do not pass through $(7,8)$?

   There are $\binom{25}{10} = 3,268,760$ such paths. To pass through $(7,8)$ one must first go from $(0,0)$ to $(7,8)$, then from $(7,8)$ to $(15,10)$. This can be done in $\binom{15}{7}\binom{10}{8} = 289,575$ ways. Hence the number of paths that DON'T pass through $(7,8)$ is the difference $2,979,185$.

4. Suppose k, n are integers satisfying $0 < k < n$. Prove that $\binom{n}{k-1}\binom{n}{k+1} \leq \binom{n}{k}\binom{n}{k}$.

   With the restrictions given, all four binomial coefficients are nonzero, so the problem is equivalent to proving that $f(n,k) \leq 1$, for $f(n,k) = \frac{\binom{n}{k-1}\binom{n}{k+1}}{\binom{n}{k}\binom{n}{k}}$. We rewrite as $f(n,k) = \frac{(n)_{k-1}}{(n)_k}\frac{(n)_{k+1}}{(n)_k}\frac{k!}{(k-1)!}\frac{k!}{(k+1)!} = \frac{1}{n-k+1}\frac{n-k}{1}\frac{k}{1}\frac{1}{k+1} = \frac{n-k}{n-k+1}\frac{k}{k+1}$, a product of two fractions each less than 1, hence itself less than 1.

5. For $m, n \in \mathbb{N}_0$, prove that $\sum_{0 \leq k \leq n} (k)_m = \frac{1}{m+1}(n+1)_{m+1}$.

   We begin with $\sum_{0 \leq k \leq n} \binom{k}{m} = \sum_{m \leq k \leq n} \binom{k}{m} = \binom{n+1}{m+1}$, which we can prove combinatorially, by induction, or recall as Thm. 4.5 in the text. Since $m \geq 0$, we may rewrite as $\sum_{0 \leq k \leq n} \frac{(k)_m}{m!} = \frac{(n+1)_{m+1}}{(m+1)!}$. Multiply both sides by $m!$ to get the desired result.
   Note the similarity between this statement and the familiar calculus statement
   $\int_0^{n+1} x^m dx = \frac{1}{m+1} x^{m+1}|_0^{n+1} = \frac{1}{m+1}(n+1)^{m+1}$. This is no coincidence, as this problem may also be solved using the fundamental theorem of difference calculus, a discrete analog of the fundamental theorem of differential calculus that you learned in your MATH 150 course.