MATH 579 Exam 3 Solutions

Part I: Let $n, k \in \mathbb{N}_0$. We choose k subsets of $[n]$, namely $S_1, S_2, \ldots, S_k$. We insist that $S_1 \cap S_2 \cap \cdots \cap S_k = \emptyset$. How many ways can this be done, as a function $f(n,k)$?

> Let $x \in [n]$. We form a bijection between (all but one of the) lists of k drawn from [2], and the membership properties of x with regards to $S_1, S_2, \ldots S_k$. If $x \in S_1$, then we set the first element of the list to be 1; if $x \notin S_1$, then we set the first element of the list to be 2. Similarly, if $x \in S_i$, we set the i^{th} element of the list to be 1; otherwise 2. The condition $S_1 \cap S_2 \cap \cdots \cap S_k = \emptyset$ is equivalent to $x \notin S_1 \cap S_2 \cap \cdots \cap S_k$; that is $(1, 1, \ldots, 1)$ is not in the range of this function. However, every other such list is possible, and lists correspond 1-1 with membership properties – that is, if x and x' had different membership properties, then they would have different corresponding lists. So, considering x alone, there are $2^k - 1$ ways to include it in the S_i's. Each other element of $[n]$ has the quantity of possible placements, and these quantities are independent of each other so the multiplication principle applies. [Note: if desired, one may biject between $f(n,k)$ and lists of length n from $[2^k - 1]$]. Thus $f(n,k) = (2^k - 1)^n$.

Part II:

1. How many four-digit positive integers are there in which all the digits are different?

   Solution 1: The first digit can be any of 9 choices (not 0). The second can be any of 9 choices (not the 1st digit). The third can be any of 8 choices (not the 1st or 2nd). The fourth can be any of 7 choices. Hence the answer is $9 \cdot 9 \cdot 8 \cdot 7 = 4,536$.
   Solution 2: We consider all l.o.d.e.'s with $n = 10, k = 4$. There are $(10)_4 = 5040$ of these. However, some begin with 0, and then have an l.o.d.e. with $n = 9, k = 3$. There are $(9)_3 = 504$ of these. Hence the answer is $5040 - 504 = 4,536$.

2. You need to visit four cities, each of them three times. How many ways can you do this if you're not allowed to start and end in the same city?

   We first ignore the restriction. This counts the number of anagrams of AAABBBCCCDDD, which is $\frac{12!}{3!3!3!3!}$. Suppose now we start and end with A. Then the middle part is an anagram of ABBBCCCDDD, of which there are $\frac{10!}{1!3!3!3!}$. But there are four such forbidden routes, hence the answer is $\frac{12!}{3!3!3!3!} - 4\frac{10!}{1!3!3!3!} = 369600 - 4(16800) = 302,400$.

3. How many permutations of $[n]$ are there such that the sum of every two consecutive elements is odd?

   The condition is satisfied if the elements alternate in parity. If n is odd, then the first and last elements (as well as every odd-indexed element) must be odd, while the even-indexed elements must be even. Hence there are

$(\frac{n+1}{2})!(\frac{n-1}{2})!$ such permutations. If n is even, then there are two mutually exclusive types of solutions: where the even-indexed elements are even, and where the odd-indexed elements are even. Hence there are $2(\frac{n}{2})!(\frac{n}{2})!$ such permutations.

4. How many ways are there to place three red, two white, and one green rook on a chessboard so that none of them attack any other?

   We first distinguish all the rooks. There are $(8)_6^2$ ways to place six distinguished nonattacking rooks on the chessboard, since the first rook has $8 \cdot 8$ choices (row and column), the second has $7 \cdot 7$ choices (row and column must differ from the first rook), etc. We now make equivalence classes on these placements, where we consider two placements identical if they differ by rearrangements of rooks of the same color. Each equivalence class has 3!2!1! elements, hence the answer is $\frac{(8)_6^2}{3!2!1!} = 33,868,800$.

5. How many solutions are there to $x_1 + x_2 + x_3 = 25$ such that $x_i \in \mathbb{N}_0$ and $x_1 \geq 4$?

   Set $y = x_1 - 4$; $x_1 = y_1 + 4$. $x_1 \geq 4$ if and only if $y \geq 0$. There is therefore a bijection between what we want and solutions to $y + x_2 + x_3 = 21$ where all three numbers are from $\mathbb{N}_0$. Now, we biject between this problem and the problem of choosing multisets with $k = 21, n = 3$. Let y denote the number of 1's chosen, x_2 denote the number of 2's chosen, and x_3 denote the number of 3's chosen. Since $k = 21$, we're choosing 21 altogether. Hence the answer is $\left(\binom{3}{21}\right) = \binom{23}{21} = \binom{23}{2} = 253$.

Exam grades: High score=104, Median score=80, Low score=52