MATH 579 Exam 2 Solutions

Part I: In the coin game, we place n coins in a row, each either heads or tails. We can remove any of the heads, leaving a gap, but when we do we flip all adjacent coins. This means at most two coins get flipped, but often fewer because gaps destroy adjacency. We keep doing this until we run out of heads; we win if all the coins are gone. For example we could play $THHH \to H \cdot TH \to H \cdot H \cdot \to H \cdot\cdot\cdot \to \cdot\cdot\cdot\cdot$ (we win), or we could play as $THHH \to TT \cdot T$ (we lose). Prove that it is possible to win if and only if the number of initial heads is odd.

> Strong induction on n. If $n = 1$, we can win if the coin is heads (one head, and 1 is odd), and we cannot otherwise (no heads, and 0 is even). For $n > 1$, if the number of heads is odd, then we remove the first head. Assume for the moment that this is not the first coin. This yields two shorter rows that each had an even number of heads (possibly zero), but then we flip one coin in each yielding an odd number of heads since even + 1 and even - 1 are both odd. Hence by strong induction we can win on each of these rows. If the first head happened to be the first coin, there is no first shorter row, but we equally win on the second shorter row by strong induction.
> Now, for $n > 1$, suppose the number of heads is even. Depending on which head we remove, we get a partition of the (even) number of heads as $\textit{even total} = \textit{left} + 1 + \textit{right}$; hence either $\textit{left}$ or $\textit{right}$ must be odd. After we flip we have an even number of heads on that side. By strong induction that shorter row cannot be won.

Part II:

1. Consider the sequence $a_0 = 1, a_{n+1} = 3a_n + 2$ (for $n \geq 0$). Prove that $a_n = 2 \cdot 3^n - 1$.

   For $n = 0$, $a_0 = 1 = 2 \cdot 3^0 - 1$. For $n > 0$, $a_n = 3a_{n-1} + 2 = 3(2 \cdot 3^{n-1} - 1) + 2 = 2 \cdot 3^n - 3 + 2 = 2 \cdot 3^n - 1$, where the first step used the recurrence relation and the second the induction hypothesis.

2. Consider the sequence $b_0 = 3.5, b_{n+1} = \sqrt{b_n + 7}$. Prove that $b_n \in (3, 4)$ for all $n \in \mathbb{N}_0$. (i.e. $3 < b_n < 4$)

   For $n = 0$, $b_0 = 3.5 \in (3, 4)$. We assume $b_n \in (3, 4)$ and rewrite the recurrence as $b_n = b_{n+1}^2 - 7$. Hence $3 < b_{n+1}^2 - 7 < 4$, so $10 < b_{n+1}^2 < 11$, so $\sqrt{10} < b_{n+1} < \sqrt{11}$ (since all $b_i > 0$). Since $\sqrt{10} \approx 3.16$ and $\sqrt{11} \approx 3.32$, we have $b_{n+1} \in (\sqrt{10}, \sqrt{11}) \subseteq (3, 4)$.

3. For all $n \in \mathbb{N}$, prove that $5 | 2^{2n-1} + 3^{2n-1}$.

   For $n = 1$, $2^{2-1} + 3^{2-1} = 5$, and $5|5$. For $n > 1$, $2^{2(n+1)-1} + 3^{2(n+1)-1} = 4 \cdot 2^{2n-1} + 9 \cdot 3^{2n-1} = 4(2^{2n-1} + 3^{2n-1}) + 5 \cdot 3^{2n-1}$. 5 divides the first term by the inductive hypothesis, and the second term since it a multiple of 5; hence 5 divides the sum.

4. For all $n \in \mathbb{N}$, prove that $3 | n^3 + 11n$.

   For $n = 1$, $1^3 + 11 = 12$, which is a multiple of 3. Considering now $n+1$, $(n+1)^3 + 11(n+1) = n^3 + 3n^2 + 3n + 1 + 11n + 11 = (n^3 + 11n) + (3n^2 + 3n + 12) = (n^3 + 11n) + 3(n^2 + n + 4)$. By the inductive hypothesis, $n^3 + 11n$ is a multiple of 3; so is $3(n^2 + n + 4)$, hence so is their sum.

5. Prove the Bernoulli inequality: if $1 + a > 0$, then $(1 + a)^n \geq 1 + na$ for all $n \in \mathbb{N}$.

   For $n = 1$, the inequality claims that $1 + a \geq 1 + a$, which is true. Otherwise we take as inductive hypothesis that $(1 + a)^n \geq 1 + na$. We multiply both sides by $1 + a$, which is positive so the direction of the inequality is unchanged, to get $(1+a)^{n+1} \geq (1+na)(1+a) = 1 + (n + 1)a + na^2 \geq 1 + (n + 1)a$, since $na^2 \geq 0$ due to $n \in \mathbb{N}$ and every square being nonnegative.

Exam grades: High score=104, Median score=84, Low score=70