

MATH 579 Spring 2009 Supplement: Recurrences

A recurrence is a sequence of numbers, defined by some positional relationship. This positional relationship is called a recurrence relation. That is, the n^{th} number is a function of the previous numbers. Some examples of recurrence relations are $a_n = 2a_{n-1}$, $b_n = b_{n-2} + 2$, $c_n = c_{n-1} + c_{n-2}$ (Fibonacci numbers if $c_1 = c_2 = 1$), $d_{n+1} = n(d_{n-1} + d_n)$ (derangements if $d_1 = 0, d_2 = 1$). To fully specify the sequence, ‘enough’ initial conditions are necessary. For example, $\{a_n\}$ requires one initial condition (e.g. $a_1 = 3$). $\{b_n\}$ requires two; $b_1 = 3$ is enough to specify all the odd terms in the sequence, but to specify the even terms we need $b_2 = 4$.

To solve a recurrence means to find a closed-form expression for the sequence, that does not depend on previous terms. Assuming you have psychic powers, the best way to solve recurrences is by guessing. A recurrence is completely specified by its initial conditions and recurrence. If you can guess the answer and show that your guess satisfies the recurrence and satisfies the initial conditions – this is enough to prove your answer.

Example 1a: $a_1 = 1, a_n = 2a_{n-1}$ ($n \geq 2$)

Guess $a_n = 2^{n-1}$. Check that $2^{1-1} = 1$, so the initial condition is satisfied. Also, $2^{n-1} = 2 \times 2^{(n-1)-1}$, so the recurrence relation is satisfied.

Much as with differential equations, recurrences fall into many types, with many different strategies for solution. A *linear* recurrence relation of *order* k may be written as $a_n = \star a_{n-1} + \star a_{n-2} + \cdots + \star a_{n-k} + \star$, where each \star is some function of n . If each \star is, in fact, a constant, we say that the recurrence has *constant coefficients*. In this section, we will only consider linear recurrences. Further, we will assume that all the coefficients (except possibly the final \star) are constants. If the final \star is identically zero (i.e. $a_n = \star a_{n-1} + \star a_{n-2} + \cdots + \star a_{n-k}$) we call the relation *homogeneous*; otherwise we call it *nonhomogeneous*. In the above examples, $a_n = 2a_{n-1}$ is first-order homogeneous with constant coefficients, $b_n = b_{n-2} + 2$ is second-order nonhomogeneous with constant coefficients, $c_n = c_{n-1} + c_{n-2}$ is second-order homogeneous with constant coefficients, and $d_{n+1} = n(d_{n-1} + d_n)$ is second-order homogeneous with nonconstant coefficients.

Homogeneous Linear Recurrence Relations with Constant Coefficients

We consider the recurrence relation $a_n = c_{n-1}a_{n-1} + c_{n-2}a_{n-2} + \cdots + c_{n-k}a_{n-k}$. Because this is homogeneous, we may multiply a solution by any constant and it will be a solution. We may also add two solutions and get a solution. In short, the set of solutions forms a linear space. This space is of dimension k , because the relation is of order k and requires k initial conditions to fully specify the recurrence. Hence, to find the general solution, we may find k linearly independent solutions, and take all their linear combinations. Caution: be sure that the k specific solutions are linearly independent.

Let’s guess that $a_n = x^n$ is a solution. We substitute into the recurrence to get $x^n =$

$c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + \dots + c_{n-k}x^{n-k}$. Dividing by x^{n-k} gives us $x^k = c_{n-1}x^{k-1} + c_{n-2}x^{k-2} + \dots + c_{n-k}$. This is known as the *characteristic equation* of the recurrence relation. It is a polynomial of degree k , and therefore by the Fundamental Theorem of Algebra has k complex roots, counted by multiplicity.

If the k roots r_1, r_2, \dots, r_k are all distinct, then $a_n = r_1^n, a_n = r_2^n, \dots, a_n = r_k^n$ are k linearly independent solutions, and therefore span the solution space. The general solution is therefore $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \dots + \alpha_k r_k^n$. The k initial conditions allow us to determine the unknown $\alpha_1, \alpha_2, \dots, \alpha_k$ for a particular solution.

If, on the other hand, a root is repeated (i.e. $r_1 = r_2$), then $a_n = r_1^n, a_n = r_2^n, \dots, a_n = r_k^n$ are *NOT* k linearly independent solutions. $\alpha_1 r_1^n + \alpha_2 r_2^n$ is a one-dimensional subspace, being equal to $\alpha_1 r_1^n$ alone, because $r_1 = r_2$. Fortunately, if a root is repeated, we have available to us additional solutions, that are linearly independent. If root r_1 has multiplicity 4, then $r_1^n, nr_1^n, n^2 r_1^n, n^3 r_1^n$ are four linearly independent solutions (this fact will not be proved). In this manner we again get k linearly independent solutions, and therefore the general solution via linear combinations.

Example 1b: $a_1 = 1, a_n = 2a_{n-1}$ ($n \geq 2$)

This has characteristic equation $x = 2$; hence the general solution is $a_n = \alpha 2^n$. Substituting $n = 1$ and using the initial conditions, we have $1 = a_1 = \alpha 2^1$. We solve to find $\alpha = 1/2$; hence the specific solution is $a_n = (1/2)2^n = 2^{n-1}$.

Example 2: $a_1 = a_2 = 1, a_n = a_{n-1} + a_{n-2}$ ($n \geq 3$) (Fibonacci numbers)

This has characteristic equation $x^2 = x + 1$, which has roots (using the quadratic formula) $r_1 = (1 + \sqrt{5})/2$ and $r_2 = (1 - \sqrt{5})/2$. Hence the general solution is $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$. We have two initial conditions: $1 = a_1 = \alpha_1 r_1 + \alpha_2 r_2, 1 = a_2 = \alpha_1 r_1^2 + \alpha_2 r_2^2$. This is a 2×2 linear system in the unknowns α_1, α_2 , with solution $\alpha_1 = \frac{1}{\sqrt{5}}, \alpha_2 = \frac{-1}{\sqrt{5}}$. Hence the specific solution is $a_n = (r_1^n - r_2^n)/\sqrt{5}$.

Example 3: $a_0 = a_2 = 1, a_1 = 0, a_3 = 2, a_n = -a_{n-1} + 3a_{n-2} + 5a_{n-3} + 2a_{n-4}$ ($n \geq 4$)

This has characteristic equation $x^4 + x^3 - 3x^2 - 5x - 2 = 0$. We find the roots by guessing small integers (the rational root theorem helps too); if we successfully guess a root r , we divide by $x - r$ using long division and continue. In this manner, we find roots -1 (multiplicity 3), and 2. Hence, the general solution is $a_n = \alpha_1(-1)^n + \alpha_2 n(-1)^n + \alpha_3 n^2(-1)^n + \alpha_4 2^n$. We now apply our initial conditions to get:

$$(n = 0) : 1 = a_0 = \alpha_1 + \alpha_4 \quad (n = 1) : 0 = a_1 = \alpha_1(-1) + \alpha_2(-1) + \alpha_3(-1) + \alpha_4 2$$

$$(n = 2) : 1 = a_2 = \alpha_1 + \alpha_2 2 + \alpha_3 4 + \alpha_4 4 \quad (n = 3) : 2 = a_3 = \alpha_1(-1) + \alpha_2(-3) + \alpha_3(-9) + \alpha_4 8$$

This is a 4×4 linear system, with solution $\alpha_1 = 7/9, \alpha_2 = -3/9, \alpha_3 = 0, \alpha_4 = 2/9$. Therefore, the specific solution is $a_n = (7/9)(-1)^n - (3n/9)(-1)^n + (2/9)2^n$.

Example 4 (Gambler's ruin): A gambler repeatedly plays a game against a casino, until one of them runs out of money. Each time the gambler has probability p of winning \$1, and probability $q = 1 - p$ of losing \$1. The gambler starts with n dollars, and the casino with $m - n$ dollars (there are m total dollars to be won). What is the probability that the gambler will run out of money before the casino?

Let a_n denote the desired probability, that the gambler is successful starting with n dollars. For the gambler to win, either (1) gambler wins first bet, and then is successful starting with $n + 1$ dollars, or (2) gambler loses first bet, and then is successful starting with $n - 1$ dollars. Therefore, this sequence satisfies the recurrence relation $a_n = pa_{n+1} + qa_{n-1}$ ($0 < n < m$), with boundary conditions $a_0 = 1, a_m = 0$. This has characteristic equation $px^2 - x + q = 0$, with roots $r_1 = 1, r_2 = q/p$. Hence the problem breaks into two cases, depending on whether $p = q$ or not.

($p \neq q$): The general solution is $a_n = \alpha 1^n + \beta r_2^n = \alpha + \beta r_2^n$. We apply the boundary conditions, to get ($n = 0$) : $1 = a_0 = \alpha + \beta$, ($n = m$) : $0 = a_m = \alpha + \beta r_2^m$. This has solution $\alpha = -r_2^m / (1 - r_2^m), \beta = 1 / (1 - r_2^m)$. Hence, the specific solution is $(-r_2^m + r_2^n) / (1 - r_2^m) = 1 - \frac{1 - r_2^n}{1 - r_2^m}$.

($p = q = 1/2$): The general solution is $a_n = \alpha 1^n + \beta n 1^n = \alpha + \beta n$. We apply the boundary conditions, to get ($n = 0$) : $1 = a_0 = \alpha$, ($n = m$) : $0 = a_m = \alpha + \beta m$. This has solution $\alpha = 1, \beta = -1/m$. Hence, the specific solution is $1 - (n/m)$.

Nonhomogeneous Linear Recurrence Relations

We want to solve the nonhomogeneous recurrence relation $a_n = c_{n-1}a_{n-1} + c_{n-2}a_{n-2} + \dots + c_{n-k}a_{n-k} + b(n)$, where $b(n)$ is a function of n . The technique to find the general solution is in two parts. First, drop the $b(n)$ term and find the general solution to the homogeneous recurrence relation. Then, find any single solution to the nonhomogeneous recurrence (under any initial/boundary conditions). The general solution to the nonhomogeneous recurrence is the sum of these two – a k -dimensional term from the homogeneous part, and a single term with no constants from the nonhomogeneous part.

Finding a particular solution is, at times, an art form. The only good way to find them is to guess and check – guess a particular solution, and see if it fits the nonhomogeneous relation. If $b(n)$ is a polynomial, it's a good idea to try guessing a polynomial of the same degree; however, if the homogeneous solution has overlap with this, then increase the degree of your guess. If $b(n)$ is an exponential, it's a good idea to try a multiple of the same exponential.

Example 5: $a_0 = 2, a_n = 2a_{n-1} + 3^n$ ($n \geq 1$)

Homogeneous version: $a_n = 2a_{n-1}$, which has characteristic equation $x = 2$ and general solution $\alpha 2^n$.

Nonhomogeneous version: Let's guess $\beta 3^n$. Plugging into the relation, we get $\beta 3^n =$

$2\beta 3^{n-1} + 3^n$. We divide both sides by 3^{n-1} to get $3\beta = 2\beta + 3$; hence $\beta = 3$. Thus 3^{n+1} is a specific solution to the original, nonhomogeneous, recurrence.

Putting them together, we find the general solution to the nonhomogeneous recurrence is $a_n = \alpha 2^n + 3^{n+1}$. We now consider the initial condition, $(n = 0) : 2 = a_0 = \alpha 2^0 + 3^1$. This has solution $\alpha = -1$, and so the specific solution is $a_n = 3^{n+1} - 2^n$.

Example 6: $a_0 = a_1 = 1, a_n = 2a_{n-1} - a_{n-2} + 5^n$ ($n \geq 2$)

Homogeneous version: $a_n = 2a_{n-1} - a_{n-2}$, which has characteristic equation $x^2 - 2x + 1 = 0$. This has a double root of 1, hence has general solution $\alpha_1 1^n + \alpha_2 n 1^n = \alpha_1 + \alpha_2 n$.

Nonhomogeneous version: Let's guess $\beta 5^n$. Plugging into the relation, we get $\beta 5^n = 2\beta 5^{n-1} - \beta 5^{n-2} + 5^n$. We divide both sides by 5^{n-2} to get $25\beta = 10\beta - \beta + 25$. This has solution $\beta = 25/16$, so a nonhomogeneous solution is $(25/16)5^n = 5^{n+2}/16$.

Putting them together, we find the general solution to the nonhomogeneous recurrence is $a_n = \alpha_1 + \alpha_2 n + 5^{n+2}/16$. Considering the initial conditions, $(n = 0) : 1 = a_0 = \alpha_1 + 25/16$, $(n = 1) : 1 = a_1 = \alpha_1 + \alpha_2 + 125/16$. This has solution $\alpha_1 = -9/16, \alpha_2 = -132/16$, and so the specific solution is $a_n = (-9 - 132n + 5^{n+2})/16$.

Example 7: $a_0 = 2, a_n = 3a_{n-1} - 4n$ ($n \geq 1$)

Homogeneous version: $a_n = 3a_{n-1}$, which has characteristic equation $x = 3$ and general solution $\alpha 3^n$.

Nonhomogeneous version: We guess a solution of $\beta_1 n + \beta_0$. Plugging into the nonhomogeneous equation, we get $(\beta_1 n + \beta_0) = 3(\beta_1(n-1) + \beta_0) - 4n$. Simplifying, we get $0 = (2\beta_1 - 4)n + (-3\beta_1 + 2\beta_0)$. If a polynomial equals zero, then each coefficient must equal zero; hence $0 = 2\beta_1 - 4$ and $0 = -3\beta_1 + 2\beta_0$. We solve this system to get $\beta_1 = 2, \beta_0 = 3$. Hence $2n + 3$ is a solution to the nonhomogeneous recurrence.

Putting them together, we find the general solution to the nonhomogeneous recurrence is $a_n = \alpha 3^n + 2n + 3$. With our initial condition, we have $(n = 0) : 2 = a_0 = \alpha 3^0 + 3$, so $\alpha = -1$. So the specific solution is $a_n = -3^n + 2n + 3$.

Example 8 (Tower of Hanoi): We have three pegs and n disks of different sizes. The disks all start on one peg arranged in order of size, and we must move them to another. We move one disk at a time, and may never put a larger disk onto a smaller. How many moves does it take?

Let a_n represent the answer. We see that $a_1 = 1$. To move the biggest disk from peg 1 to peg 2, all the smaller disks must be in a single stack, on peg 3. Therefore, the solution must contain three steps: First, move the $n - 1$ smaller disks from peg 1 to peg 3, then move the largest disk from peg 1 to peg 2, then move the $n - 1$ smaller disks back onto the largest disk from peg 3 to peg 2. Hence, $a_n = a_{n-1} + 1 + a_{n-1} = 2a_{n-1} + 1$.

The homogeneous recurrence is again $a_n = 2a_{n-1}$ with general solution $\alpha 2^n$. To find a specific

solution to the nonhomogeneous recurrence, consider a constant (0-th degree) polynomial in n , say β . Plugging into the nonhomogeneous equation, we get $\beta = 2\beta + 1$; we solve this to get $\beta = -1$. Hence the general solution to the nonhomogeneous relation is $a_n = \alpha 2^n - 1$. Our initial conditions tell us $1 = a_1 = \alpha 2^1 - 1$; hence $\alpha = 1$ and our specific solution is $a_n = 2^n - 1$.

Example 9 (Gambler's ruin revisited): Consider the gambler of example 4. What is the expected number of games played until either the gambler or casino is ruined?

Let a_n denote the desired answer (when the gambler starts with $\$n$). If the gambler wins, then the expected number of games is one more than the expected number of games, had the gambler started with $\$(n + 1)$. If the gambler loses, then the expected number of games is one more than the expected number of games, had the gambler started with $\$(n - 1)$. Hence we get the relation $a_n = p(a_{n+1} + 1) + q(a_{n-1} + 1)$ ($0 < n < m$). We have boundary conditions $0 = a_0 = a_m$, and may rewrite the relation as $pa_{n+1} = a_n - qa_{n-1} - 1$. The homogeneous recurrence has the familiar characteristic equation $px^2 - x + q = 0$; once again the problem splits into cases based on whether $q = p$.

($p \neq q$): The homogeneous general solution is $\alpha + \beta r_2^n$ (recall that $r_2 = q/p$). If we try to guess a 0-th degree polynomial solution to the nonhomogeneous recurrence, we will find no luck (try it and see). The reason is that all 0-th degree polynomials are already solutions of the homogeneous recurrence, and so none of them could ever solve the nonhomogeneous recurrence.

Instead let's try a first-degree polynomial $c_1 n + c_0$. We plug into the nonhomogeneous equation to get $p(c_1(n + 1) + c_0) = c_1 n + c_0 - q(c_1(n - 1) + c_0) - 1$. We collect terms to get $n(pc_1 - c_1 + qc_1) + (pc_1 + pc_0 - c_0 - qc_1 + qc_0 + 1) = 0$. The first coefficient is zero already, and the second coefficient simplifies to $(p - q)c_1 + 1 = 0$; hence $c_1 = -1/(p - q)$, and we may as well take $c_0 = 0$ although the choice is arbitrary (in fact, we could have known this since all constants are part of the homogeneous solution). Therefore, the general nonhomogeneous solution is $a_n = \alpha + \beta r_2^n - n/(p - q)$. For the particular solution, we take $0 = a_0 = \alpha + \beta, 0 = a_m = \alpha + \beta r_2^m - m/(p - q)$. This has solution $\beta = \frac{m}{(1-2p)(1-r_2^m)}, \alpha = -\beta$. We plug these into the general solution, to find $a_n = \left(n - m \frac{1-r_2^n}{1-r_2^m}\right)/(1 - 2p)$.

($p = q = 1/2$): The homogeneous general solution is $\alpha + \beta n$. We won't get very far trying low-degree polynomials, since they are all part of the homogeneous solution. So, let's try cn^2 . We plug into the nonhomogeneous equation to get $pc(n + 1)^2 = cn^2 - qc(n - 1)^2 - 1$. We rewrite to get $n^2(pc - c + qc) + n(2pc - 2qc) + (pc + qc + 1) = 0$. Since $p = q = 1/2$, the first two coefficients are zero already, and the last is zero when $c = -1$. Hence the general nonhomogeneous solution is $a_n = \alpha + \beta n - n^2$. For the specific solution, we take $0 = a_0 = \alpha, 0 = a_m = \alpha + \beta m - m^2$. This has solution $\alpha = 0, \beta = m$. Hence, the specific solution is $a_n = mn - n^2 = n(m - n)$.

Exercises

Solve the following recurrence relations.

1. $a_0 = a_1 = 2, a_n = -2a_{n-1} - a_{n-2} (n \geq 2)$
2. $a_0 = 0, a_1 = 1, a_n = 4a_{n-2} (n \geq 2)$
3. $a_0 = 2, a_1 = -4, a_2 = 26, a_n = a_{n-1} + 8a_{n-2} - 12a_{n-3} (n \geq 3)$
4. $a_0 = a_1 = a_2 = 0, a_n = 9a_{n-1} - 27a_{n-2} + 27a_{n-3} (n \geq 3)$
5. $a_0 = a_1 = 0, a_n = a_{n-1} + 2a_{n-2} + 3 (n \geq 2)$
6. $a_0 = a_1 = 0, a_n = a_{n-1} + 2a_{n-2} + n (n \geq 2)$
7. $a_0 = a_1 = 0, a_n = a_{n-1} + 2a_{n-2} + e^n (n \geq 2)$
8. What is the maximum number of regions we can divide the plane into, using n lines?
9. Let a_n be the number of n -digit nonnegative integers in which no three consecutive digits are the same. Justify that $a_{n+2} = 9a_{n+1} + 9a_n$, then find a_n .
10. Let a_n be the number of ways to color the squares of a $1 \times n$ chessboard using the colors red, white, and blue, so that no two red squares are adjacent.
11. Let a_n be the number of ways to color the squares of a $1 \times n$ chessboard using the colors red, white, and blue, so that no red square is adjacent to a white square.
12. Let a_n be the number of ways to color the squares of a $1 \times n$ chessboard using the colors red, white, and blue, so that the specific sequence red-white-blue does not occur.
13. Let a_n be the number of ways to climb a flight of n stairs, when each of your steps may move you one, two, or three steps higher.
14. Codewords (strings) from the alphabet $\{0, 1, 2, 3\}$ are called *legitimate* if they have an even number of 0's. How many legitimate codewords are there, of length k ?