MATH 579 Exam 7 Solutions

1. How many integers in $[1, 900000]$ are relatively prime to 900000?

   $900000 = 3^2 2^5 5^5$, so $\phi(900000) = \phi(3^2)\phi(2^5)\phi(5^5) = (3^2 - 3)(2^5 - 2^4)(5^5 - 5^4) = 240000$.

   Or, directly, let P_1, P_2, P_3 denote the property of being divisble by $2, 3, 5$ respectively. The desired quantity is $|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| = 900K - \lfloor \frac{900K}{2} \rfloor - \lfloor \frac{900K}{3} \rfloor - \lfloor \frac{900K}{5} \rfloor + \lfloor \frac{900K}{6} \rfloor + \lfloor \frac{900K}{10} \rfloor + \lfloor \frac{900K}{15} \rfloor - \lfloor \frac{900K}{30} \rfloor = 240000$

2. How many four-digit integers are not divisible by 6, 7, or 8?

   Let P_1, P_2, P_3 denote the property of being divisble by $6, 7, 8$ respectively. Recall that $P_1 \cap P_3$ is the property of being divisible by $\text{LCM}(6, 8) = 24$. The number of desired integers in [1,9999] is $9999 - \lfloor \frac{9999}{6} \rfloor - \lfloor \frac{9999}{7} \rfloor - \lfloor \frac{9999}{8} \rfloor + \lfloor \frac{9999}{42} \rfloor + \lfloor \frac{9999}{24} \rfloor + \lfloor \frac{9999}{56} \rfloor - \lfloor \frac{9999}{168} \rfloor = 6429$. The number of desired integers in [1,999] is $999 - \lfloor \frac{999}{6} \rfloor - \lfloor \frac{999}{7} \rfloor - \lfloor \frac{999}{8} \rfloor + \lfloor \frac{999}{42} \rfloor + \lfloor \frac{999}{24} \rfloor + \lfloor \frac{999}{56} \rfloor - \lfloor \frac{999}{168} \rfloor = 643$. Hence the number of desired integers in [1000,9999] is $6429 - 643 = 5786$.

3. How many n-permutations are there with exactly one cycle of length one?

   Suppose first that the length-one cycle is (1). The remaining $n - 1$ elements can be any permutation where no element is sent to itself (otherwise there would be a second length-one cycle), a derangement. Hence there are D_{n-1} such permutations, with (1) the length-one cycle. However, there were n choices for the length-one cycle, so the answer is nD_{n-1}.

4. How many 10-permutations are there with exactly one descent?

   Say the descent is at position i, with $1 \le i \le 9$. We partition [10] into one part of size i, and one part of the rest. There are $\binom{10}{i}$ ways to do this. There is one way to build a permutation of the desired type from these parts: each part must be in increasing order or there would be a second descent. Further, exactly one of these $\binom{10}{i}$ is forbidden: if the i components of the first part are exactly $[i]$ there are no descents at all. Thus the answer is $\sum_{i \in [1,9]} \binom{10}{i} - 1 = -11 + \sum_{i \in [0,10]} \binom{10}{i} = -11 + 2^{10} = 1013$.

5. How many solutions are there to $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 8$ where $0 \le x_i \le i$?

   Set property P_i to be that $x_i \ge i + 1$. With no restrictions, we put 8 identical balls into 6 different bins $\left(\!\binom{6}{8}\!\right)$. With P_4, we must have at least five balls in bin 4; remove them and we put 3 identical balls into 6 different bins $\left(\!\binom{6}{3}\!\right)$. With $P_1 \cap P_2$ we must have at least two balls in bin 1 and three balls in bin 2; remove them and we put 3 identical balls into the bins $\left(\!\binom{6}{3}\!\right)$.

   Putting it all together, we get $\left(\!\binom{6}{8}\!\right) - \left(\!\binom{6}{6}\!\right) - \left(\!\binom{6}{5}\!\right) - \left(\!\binom{6}{4}\!\right) - \left(\!\binom{6}{3}\!\right) - \left(\!\binom{6}{2}\!\right) - \left(\!\binom{6}{1}\!\right) + \left(\!\binom{6}{3}\!\right)_{(P_1 \cap P_2)} + \left(\!\binom{6}{2}\!\right)_{(P_1 \cap P_3)} + \left(\!\binom{6}{1}\!\right)_{(P_1 \cap P_4)} + \left(\!\binom{6}{0}\!\right)_{(P_1 \cap P_5)} + \left(\!\binom{6}{1}\!\right)_{(P_2 \cap P_3)} + \left(\!\binom{6}{0}\!\right)_{(P_2 \cap P_4)} = 455$. No other combinations are possible.

6. How many ways are there to place five (identical) nonattacking rooks on a 5×5 chessboard, with no rooks on the diagonal?

   There must be one rook on each row. Let P_i denote the property that the rook on row i is on the diagonal (column i), and let A_i denote the set of placements that have property P_i. With no restrictions, there are 5! placements (5 choices for rook on row 1, then 4 choices for rook on row 2, etc.). $|A_1| = 4!$, since after placing the rook in the first row in its required place, there are four choices for the next rook, three for the following, etc. Similarly, $|A_2| = |A_3| = |A_4| = |A_5| = 4!$. Hence $\sum |A_i| = \binom{5}{1} 4!$. $|A_1 \cap A_2| = 3!$, since after placing the two rooks perforce, we place the three remaining rooks. Hence $\sum |A_i \cap A_j| = \binom{5}{2} 3!$. Continuing similarly, we have $\binom{5}{0} 5! - \binom{5}{1} 4! + \binom{5}{2} 3! - \binom{5}{3} 2! + \binom{5}{4} 1! - \binom{5}{5} 0! = 44$.

Exam results: High score=100, Median score=77, Low score=60 (before any extra credit)