

## MATH 579 Exam 6 Solutions

1. Consider the permutation on  $[4]$  given by  $f(1) = 4, f(2) = 1, f(3) = 3, f(4) = 2$ . Write this in two-line notation, one-line notation, canonical cycle form, and as a directed graph.

$$\left( \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{array} \right), 4132, (3)(421), \begin{array}{ccc} 1 & & 3 \\ \uparrow & \searrow & \uparrow \\ & & 2 \leftarrow 4 \end{array}$$

2. For all  $n \in \mathbb{N}$ , prove that the number of partitions of  $n$  into no more than  $k$  parts is equal to the number of partitions of  $n + 2k$  into exactly  $k$  parts, each of size at least 2.

We establish a bijection between these sets. Given a Ferrers diagram of  $n$  into at most  $k$  parts, we add 2 columns of length  $k$  to the left. This gives a diagram of  $n + 2k$  into exactly  $k$  parts, each of size at least 2. Similarly, given one of the latter diagrams, we remove the leftmost 2 columns to get a diagram of  $n$  into at most  $k$  parts.

3. Consider the permutations on  $[8]$  given by  $\pi : (51234)(876)$ ,  $\sigma : (2)(31)(745)(86)$ . Calculate  $\pi \circ \sigma \circ \pi^{-1}$ .

Note that  $\pi^{-1} = (54321)(867)$ . We must determine how  $\pi \circ \sigma \circ \pi^{-1}$  acts on each of the eight elements.  $1 \rightarrow 5 \rightarrow 7 \rightarrow 6$ ,  $2 \rightarrow 1 \rightarrow 3 \rightarrow 4$ ,  $3 \rightarrow 2 \rightarrow 2 \rightarrow 3$ ,  $4 \rightarrow 3 \rightarrow 1 \rightarrow 2$ ,  $5 \rightarrow 4 \rightarrow 5 \rightarrow 1$ ,  $6 \rightarrow 7 \rightarrow 4 \rightarrow 5$ ,  $7 \rightarrow 8 \rightarrow 6 \rightarrow 8$ ,  $8 \rightarrow 6 \rightarrow 8 \rightarrow 7$ . Putting it together, in canonical notation we get  $(3)(42)(651)(87)$ .

Note: Those studying permutations in depth call this operation ‘conjugation’ and write it as  $\sigma^\pi$ . It turns out that one can find its result by applying  $\pi$  to each element of  $\sigma$ , i.e.  $\sigma^\pi = (\pi(2))(\pi(3)\pi(1))(\pi(7)\pi(4)\pi(5))(\pi(8)\pi(6)) = (3)(42)(651)(78)$ .

4. Find all self-conjugate partitions of 20.

We consider hooks in these partitions. Each hook contains an odd number of boxes, hence there must be either 2 or 4 hooks since the sum is 20 and the hooks must have different sizes. If 2 hooks, they can be of length  $19+1:(10,2,1,1,1,1,1,1,1,1)$   $17+3:(9,3,2,1,1,1,1,1,1)$   $15+5:(8,4,2,2,1,1,1,1)$   $13+7:(7,5,2,2,2,1,1)$   $11+9:(6,6,2,2,2,2)$ . If 4 hooks, the only possibilities are  $11+5+3+1:(6,4,4,4,1,1)$  and  $9+7+3+1:(5,5,4,4,2)$ . There are thus seven such partitions.

5. For all  $n \in \mathbb{N}$ , prove that  $p(n)^2 < p(n^2 + 2n)$ .

We provide an injective function from pairs of partitions of  $n$  into partitions of  $n^2 + 2n$ . Given any two partitions of  $n$ , let  $A$  denote the Ferrers diagram of one, and  $B$  denote the Ferrers diagram of the other. We take an  $n \times n$  square and glue  $A$  to the east side and  $B$  to the south side:  $\square_B^A$ . This gives a valid Ferrers diagram for a partition of  $n^2 + 2n$ ; if we change  $A$  or  $B$  the

result will be a different partition of  $n^2 + 2n$ . Note that this function is not surjective; for example, it does not produce any partitions of  $n^2 + 2n$  into fewer than  $n$  parts. This establishes the  $<$  (as opposed to  $\leq$ ).

6. Let  $X$  be the set of 19 students enrolled in this course, and let  $Y = \{A, B, C, D, F\}$ . I want to count how many ways to assign grades (from  $Y$ ) to you all. Each of the following is WRONG; none of them count the desired quantity. For each, explain what it actually *IS* counting, including at least one example, and how this is different from what I'm trying to count. Finally, give the right answer. You need not calculate or simplify anything.

$\binom{19}{5}$  gives out indistinguishable grades exactly once; a given student may receive more than one. For example,  $s_1$  gets four grades and  $s_2$  gets one grade. Among other problems, this does not give every student a grade.  $\binom{5}{19}$  does assign 19 grades from among the five possible. However it does not distinguish who gets which grades; it is just a grade distribution. For example,  $A$  is given out 18 times,  $B$  is given out once.  $S(19, 5)$  separates the students into five categories; however it does not assign grades to the categories. For example,  $\{s_1\}\{s_2\}\{s_3, s_4\}\{s_5\}\{s_5, \dots, s_{19}\}$ .  $5!S(19, 5)$  is *almost* right. It separates the students into five nonempty categories, and assigns grades to each. However, it insists that all five grades must be awarded. For example,  $F : \{s_1\}, D : \{s_2\}, C : \{s_3, s_4\}, B : \{s_5\}, A : \{s_6, \dots, s_{19}\}$ .  $(19)_5$  gives out each of the grades exactly once, to distinct students. However, this leaves twelve students without grades. For example,  $F \rightarrow s_1, D \rightarrow s_2, C \rightarrow s_3, B \rightarrow s_4, A \rightarrow s_8$ .  $p_5(19)$  captures how many grade distributions are possible, assuming each of the five grades are awarded. However this does not distinguish who gets which grade (and insists all five grades are awarded at least once). For example,  $10 + 4 + 3 + 1 + 1$ . The right answer is  $5^{19}$ ; five choices for student  $s_1$ , five choices for student  $s_2$ , etc.

Many of you mimicked the solution from the exam from two years ago. There are several important differences between that problem and this one. Two years ago, the question was about *how* the grades are assigned (e.g. voting). This issue is immaterial here; this question is not about how assignments are made, but about *how many* assignments are possible. An answer like 'grades are not assigned by voting' is not correct; a correct answer is like 'this assignment does not distinguish among the students'. Also important to evaluating your solution is an example of what you're counting. That is the difference between demonstrating vague and concrete understanding of what is being counted.

Exam results: High score=90, Median score=80, Low score=58 (before any extra credit)