Math 579 Exam 9 Solutions

1. Solve the following recurrence. $a_n = 6a_{n-1} - 9a_{n-2}$, $a_0 = 0, a_1 = 1$.

   This homogeneous recurrence has characteristic equation $x^2 - 6x + 9 = 0$, which has a double root of $x = 3$. Hence, the general solution is $a_n = \alpha 3^n + \beta n 3^n$. The initial conditions give us $0 = a_0 = \alpha + 0, 1 = a_1 = \alpha 3 + \beta 3$, which has solution $\alpha = 0, \beta = 1/3$. Hence, the solution is $a_n = n3^{n-1}$.

2. Solve the following recurrence. $a_n = 3a_{n-1} + 2n + 1$, $a_0 = 0$.

   The homogeneous version has general solution $a_n = \alpha 3^n$. For the nonhomogeneous version, try $a_n = An + B$. We have $An + B = 3(A(n-1) + B) + 2n + 1$. We rearrange to $n(3A+2-A)+(-3A+3B+1-B) = 0$; hence $3A+2-A = 0, -3A+3B+1-B = 0$. This has solution $A = -1, B = -2$. Hence the general nonhomogeneous solution is $a_n = \alpha 3^n - n - 2$. We apply the initial condition to get $0 = a_0 = \alpha 3^0 - 0 - 2$. Hence $\alpha = 2$ and the solution is $a_n = 2(3^n) - n - 2$.

3. A gambler repeatedly plays a game against a casino, until one of them runs out of money. Each time, the gambler has probability s of nothing happening, probability p of winning \$1, and probability q of losing \$1, with $s + p + q = 1$. The gambler starts with n dollars, and the casino with $m - n$ dollars. What is the probability that the gambler will run out of money before the casino?

   This is similar to Example 4 in the handout. The conditions give us the recurrence relation $a_n = sa_n + pa_{n+1} + qa_{n-1}$, with characteristic equation $px^2 + (s-1)x + q = 0$. However, this has (surprisingly) the same roots as before, namely 1 and q/p. Therefore, the rest of Example 4 applies verbatim, giving solutions $1 - \frac{1-(q/p)^n}{1-(q/p)^m}$ for $p \neq q$ and $1 - (n/m)$ for $p = q$.

4. Solve the following recurrence. $a_n = a_{n-1} + a_{n-2} - a_{n-3} + 2$, $a_0 = 4, a_1 = 0, a_2 = 5$.

   The characteristic equation of the homogeneous recurrence is $x^3 - x^2 - x + 1 = 0$, which factors as $(x-1)^2(x+1) = 0$. Hence the general homogeneous solution is $a_n = \alpha + \beta n + \gamma(-1)^n$. For the particular nonhomogeneous solution, we note that polynomials of degree 0 and 1 are already represented in the homogeneous solution set. Hence we try $a_n = An^2$. $An^2 = A(n-1)^2 + A(n-2)^2 - A(n-3)^2 + 2 = A(n^2-2n+1)+A(n^2-4n+4)-A(n^2-6n+9)+2 = An^2-4A+2$. Hence $A = 1/2$, and the general nonhomogeneous solution is $a_n = \alpha + \beta n + \gamma(-1)^n + n^2/2$. We apply our initial conditions to get $4 = a_0 = \alpha + \gamma, 0 = a_1 = \alpha + \beta - \gamma + \frac{1}{2}, 5 = a_2 = \alpha + 2\beta + \gamma + 2$. This has solution $\alpha = \gamma = 2, \beta = -1/2$, and thus the solution is $a_n = 2+2(-1)^n+(n^2-n)/2$.

5. Let a_n represent the maximum number of regions we can divide the plane into with n lines. Find and solve a recurrence for a_n.

   We note that $a_1 = 2, a_2 = 4$. Consider adding the n^{th} line. It can cross at most $n-1$ lines (each of the previous lines once). Hence, it can cross at most n regions (since to go from one region to another it must cross a line). Since it crosses at most n regions, it can create at most n new regions. Conversely, we can ensure that it DOES create n new regions, by making it not parallel to any of the other lines and not pass through any intersection point. Hence $a_n = a_{n-1} + n$. This has homogeneous solution $a_n = \alpha(1)^n = \alpha$. For the nonhomogeneous version, we guess $a_n = An^2 + Bn$. We have $An^2 + Bn = A(n-1)^2 + B(n-1) + n$. We solve to find $A = B = 1/2$. Hence $a_n = \alpha + (n + n^2)/2$ is the general nonhomogeneous solution. Our initial condition gives us $2 = a_1 = \alpha + (1 + 1^2)/2$, so $\alpha = 1$ and the solution is $a_n = (n^2 + n + 2)/2$.

Part II. Let $a_0 = A, a_1 = B$, $a_n = \frac{1+a_{n-1}}{a_{n-2}}$ for $n > 1$. Assume that $a_n \neq 0$ for all n, so we never divide by zero. Calculate and simplify a_2, a_3, a_4, a_5, a_6. Look for a pattern, and use it to find a_n.

$a_2 = \frac{1+B}{A}, a_3 = \frac{A+B+1}{AB}, a_4 = \frac{A+1}{B}, a_5 = A, a_6 = B$, and then it all begins again (this sequence is periodic).

$$\text{Hence } a_n = \begin{cases} A & n = 5k, k \in \mathbb{N} \\ B & n = 5k+1, k \in \mathbb{N} \\ (1+B)/A & n = 5k+2, k \in \mathbb{N} \\ (1+A+B)/AB & n = 5k+3, k \in \mathbb{N} \\ (1+A)/B & n = 5k+4, k \in \mathbb{N} \end{cases}$$

Exam statistics: Low grade=32(D); Median grade=41(B); High grade=49(A)