Math 579 Exam 3 Solutions

1. Our class has 18 students, 13 males and 5 females. How many ways are there to form a study group of 4 students that contains at least one male and at least one female?

   Strategy: Number of study groups without regard to gender = x + (number of all-male study groups) + (number of all-female study groups). We want to find x; to do so, we find all three other numbers. $\binom{18}{4} = x + \binom{13}{4} + \binom{5}{4}$; hence $x = \binom{18}{4} - \binom{13}{4} - \binom{5}{4} =$ 3,060−715−5=2,340.

2. In how many ways can we place three red rooks, two black rooks, and one white rook on an ordinary 8×8 chessboard so that no two rooks attack each other?

   The six rooks will occupy six rows and six columns; these may be chosen in $\binom{8}{6}\binom{8}{6}$ different ways. Within those six rows and columns, we choose the six positions for the rooks. This can be done in 6! ways. Among these six positions, we choose three of them to receive red rooks, two to receive black rooks, and one to receive a white rook. This can be done in $\binom{6}{3\ 2\ 1}$ ways. Putting it all together, the answer is $\binom{8}{6}\binom{8}{6}6!\binom{6}{3\ 2\ 1} =$ 33,868,800.

3. Andy and Brenda are playing a game with five unusual dice, each of which has eight equally probable sides (numbered 1,2,3,4,5,6,7,8). They roll the five dice. If at least one of the dice shows an 8, then Andy wins (otherwise Brenda wins). Who is more likely to win?

   For simplicity, assume the five dice are distinguishable (different colors). There are $8^5 =$ 32,768 possible, equally likely, outcomes from rolling the dice. Brenda wins if no 8 appears; there are $7^5 =$ 16,807 ways for her to win. Therefore, there are 32,768−16,807=15,961 ways for Andy to win. Hence, Brenda is (slightly) more likely to win. More precisely, Brenda will win 16,807 times out of 32,768, which is approximately 51.3% of the time.

4. How many six-digit positive integers are there that contain the digit 0 and are divisible by 9?

   There are 900,000= 9×10^5 six-digit positive integers. One-ninth of them, or 100,000, are divisible by 9. Strategy: 100,000 = x + (number of six-digit positive integers that are divisible by 9 and do NOT contain the digit 0). To count the latter, there are nine choices for each of the first five digits (0 is forbidden). The last digit, however, is not freely chosen; it must sum with all the others to be a multiple of 9. If the others sum to 1 (modulo 9), then the last digit must be 8. If the others sum to 2 (modulo 9), then the last digit must be 7. And so on; the only interesting case is if the others sum to 0 (modulo 9). Then, the last digit could be 0 or 9; however 0 is forbidden, so the last digit must be 9. Hence, regardless of the first five digits, the last digit is determined uniquely. So, $x =$100,000$-9^5 \times 1 =$40,951.

5. How many six-digit positive integers are there that contain the digit 1 and whose digits are all different?

   Solution 1: We begin by choosing the five digits other than 1. There are $\binom{9}{5} = 126$ ways to do this. Now that we have our six digits, there are $6! = 720$ ways to arrange them; this gives $720 \times 126 = 90{,}720$. However, some of these start with 0. How many? The number of five-digit numbers with all different digits, that contain 1, and do not contain 0. Let's choose four digits other than 1 and 0; there are $\binom{8}{4} = 70$ ways to do this. Now that we have our five digits, there are $5! = 120$ ways of arranging them; this gives $70 \times 120 = 8{,}400$. Hence, the solution to the problem is $90{,}720-8{,}400=82{,}320$.

   Solution 2: Let's count separately integers that contain 0 and those that do not. Those that do not contain a 0, have five digits other than 0,1; there are $\binom{8}{5} = 56$ ways to do this. There are $6! = 720$ ways to arrange them; this gives $56 \times 720 = 40{,}320$ zero-free integers. Those that DO contain a 0, have four digits other than 0,1; there are $\binom{8}{4} = 70$ ways to do this. Now that we have selected our six digits, there are 5 choices for the first digit (not 0). There are 5 choices for the second digit (since we have used a digit already). There are 4 choices for the third digit, and so on. Hence, there are $5 \times 5 \times 4 \times 3 \times 2 \times 1 = 600$ ways to arrange these six digits; this gives $70 \times 600 = 42{,}000$ zero-containing integers. Putting these together gives $40{,}320+42{,}000 = 82{,}320$.

Part II. How many six-digit positive integers are there in which the sum of the digits is ≥ 5?

From 900,000 six-digit positive integers we take away any whose digit sum is < 5.
Digit sum 1: 100,000 only (1)
Digit sum 2: 200,000; {110,000; 101,000; 100,100; 100,010; 100,001} (6)
Note that the set denoted by {} has $\binom{5}{1}$ elements, since there are this many ways to choose one non-leading digit to be 1 (the remaining non-leading digits are 0).
Digit sum 3: 300,000; $\binom{5}{1}$ that start with 2 and have a 1 somewhere else, such as 201,000; $\binom{5}{1}$ that start with 1 and have a 2 somewhere else, such as 102,000; $\binom{5}{2}$ that start with a 1 and have two 1's somewhere else, such as 101,001. (21)
Digit sum 4: 400,000; $\binom{5}{1}$ that start with 3 and have a 1 somewhere else, such as 300,100; $\binom{5}{1}$ that start with 1 and have a 3 somewhere else, such as 100,003; $\binom{5}{1}$ that start with a 2 and have a 2 somewhere else, such as 220,000; $\binom{5}{2}$ that start with a 2 and have two 1's somewhere else, such as 210,010; 5×4 that start with a 1 and have both a 1 and a 2 somewhere else, such as 120,010; $\binom{5}{3}$ that start with a 1 and have three more 1's somewhere else, such as 101,110. (56)

There are $1+6+21+56 = 84$ six-digit numbers whose digit sum is less than 5; hence there are $900{,}000-84 = 899{,}916$ six-digit numbers whose digit sum is at least 5.

These numbers (1,6,21,56), as well as the next ones, can be found on the wonderful website `http://www.research.att.com/~njas/sequences/A090581` wherein also can be found just about any sequence of integers that have been studied.

Exam statistics: Low grade=31(D); Median grade=41.5(B); High grade=51(A+)