

## Math 579 Exam 1 Solutions

1. Find all triples of distinct positive integers  $a, b, c$  such that  $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$ .

For convenience, assume  $a < b < c$ , and let  $X = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ . If  $a = 1$ , then  $X > 1$ ; if  $a \geq 3$ , then  $X \leq \frac{1}{3} + \frac{1}{4} + \frac{1}{5} < 1$ . Hence  $a = 2$ . If  $b > 3$  then  $X \leq \frac{1}{2} + \frac{1}{4} + \frac{1}{5} < 1$ . Hence  $b = 3$ ; so  $a = 2, b = 3, c = 6$  is the only solution, up to relabeling the letters.

2. Prove that among 51 integers, there are always two so that either their sum or their difference is a multiple of 99.

Consider the pigeonholes  $\{0\}, \{1, 98\}, \{2, 97\}, \dots, \{49, 50\}$ . Into these 50 pigeonholes place the 51 remainders (upon division by 99) of the given numbers. By the PHP, there must be a collision. Hence either two integers have the same remainder (in which case their difference is a multiple of 99), or they have remainders whose sum is 99 (in which case their sum is a multiple of 99).

3. Prove that among 1001 distinct integers chosen from  $[1, 2000]$ , there are always two that are relatively prime (have greatest common divisor 1).

Consider the pigeonholes  $\{1, 2\}, \{3, 4\}, \{5, 6\}, \dots, \{1999, 2000\}$ . Into these 1000 pigeonholes place the 1001 integers. By the PHP, there must be a collision. Since the integers are distinct, two of them must be consecutive. However, any two consecutive integers are relatively prime.

4. Prove that the sequence  $2006, 20062006, 200620062006, \dots$  has an element that is a multiple of 2007.

Consider the remainders of these numbers upon division by 2007. By the PHP, the first 2008 of them must have a collision. Hence, the difference between two of these numbers is a multiple of 2007. However, this difference is itself one of these numbers, times a power of ten. However, 2007 has no prime factors in common with ten, hence 2007 must divide one of these numbers.

5. A thousand pennies are initially divided into four piles. They are then rearranged into six piles. Prove that at least three pennies end up in a smaller pile, and give an example in which exactly three pennies do so.

Just for fun, let's prove the more general statement that with any sufficient, number of coins going from  $n$  to  $n + 2$  piles will result in at least three pennies moving to a smaller pile. We will prove this with induction; this will segue nicely into Chapter 2. Many other proofs are possible.

Base case:  $n = 1$ . There must be at least three coins (in order to form three piles), and each coin will end up in a smaller pile.

Inductive case:  $n > 1$ . Let  $X$  denote the size of the biggest pile initially, and  $Y$  denote the size of the biggest pile afterward. If  $X = Y$ , then take these two biggest piles away, and consider the remaining piles. By induction, there must be at least three coins that

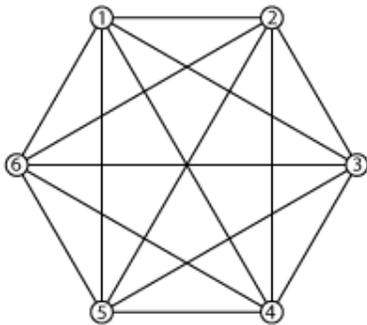
moved to a smaller pile, so we are done. If  $X > Y$ , then each coin in  $X$  moved to a smaller pile; if  $X \geq 3$ , we are done; otherwise the initial piles were all of size 2 and 1, and all the new piles are of size 1. Hence there must have been exactly two piles of size 2, which were split in half to build the two new piles; there were therefore four coins that moved to a smaller pile.

Finally, we consider the case  $X < Y$ . Take these two biggest piles away, and temporarily take  $Y - X$  extra coins from anywhere in the first configuration (so long as there are still  $n - 1$  piles). By induction, there must be at least three coins that moved to a smaller pile. Now, put the  $Y - X$  coins back. This serves only to make the original piles larger; any coin that moved to a smaller pile before has still moved to a smaller pile.

Example:  $(995, 3, 1, 1) \rightarrow (995, 1, 1, 1, 1)$

Part II: Suppose that exactly six students, including you, are enrolled in MATH 579. Assume that each pair of students are either friends or enemies. Prove that either:

1. There are three people in class that are mutual friends (all friends with each other); or
2. There are three people in class that are mutual enemies (all enemies with each other).



Alternate formulation: Prove that any red/blue coloring of the fifteen edges at left must yield either a red triangle or a blue triangle. (blue means friends, red means enemies)

Consider the five people in relation to you. They are in two categories, hence at least  $\lceil \frac{5}{2} \rceil = 3$  are in the same category. Suppose that category is “friends”; that is, you have at least three friends. If any two of your friends are friends with each other, then that forms a group of three mutual friends (you are in the group). If, however, no two of your friends are friends with each other, then that forms a group of three mutual enemies (you are not in the group).

The other case is very similar – you have at least three enemies. If any two of these enemies are enemies with each other, then that forms a group of three mutual enemies (you are in the group). If, however, no two of your enemies are enemies with each other, then that forms a group of three mutual friends (you are not in the group).

This problem is very famous, and is the tip of a large iceberg known as Ramsey theory.

Exam statistics: Low grade=31(D); Median grade=39(C+); High grade=52(A+)  
(of course, these are subject to change due to extra credit)