MATH 579: Combinatorics

Exam 4 Solutions

1. Use methods of difference calculus to compute $\sum_{i=1}^{25} i^4$.

We first translate $\sum_{i=1}^{25} i^4 = \sum_1^{26} x^4 \delta x$. Now, we compute Stirling numbers of the second kind to find $x^4 = S(4,4)x^{\underline{4}} + S(4,3)x^{\underline{3}} + S(4,2)x^{\underline{2}} + S(4,1)x^{\underline{1}} = x^{\underline{4}} + 6x^{\underline{3}} + 7x^{\underline{2}} + x^{\underline{1}}$. Hence our sum is $\sum_1^{26} x^{\underline{4}} + 6x^{\underline{3}} + 7x^{\underline{2}} + x^{\underline{1}}\delta x = \frac{1}{5}x^{\underline{5}} + \frac{6}{4}x^{\underline{4}} + \frac{7}{3}x^{\underline{3}} + \frac{1}{2}x^{\underline{2}}|_1^{26} = \frac{1}{5}26^{\underline{5}} + \frac{6}{4}26^{\underline{4}} + \frac{7}{3}26^{\underline{3}} + \frac{1}{2}26^{\underline{2}} - (\frac{1}{5}1^{\underline{5}} + \frac{6}{4}1^{\underline{4}} + \frac{7}{3}1^{\underline{3}} + \frac{1}{2}1^{\underline{2}}) = 2,153,645$.

2. Let u, v be functions from $\mathbb{Z}$ to $\mathbb{R}$. Prove that $\Delta(uv) = u\Delta v + Ev\Delta u$.

We calculate $Ev\Delta u + u\Delta v = v(x+1)\Delta u + u(x)\Delta v = v(x+1)(u(x+1) - u(x)) + u(x)(v(x+1) - v(x)) = u(x+1)v(x+1) - u(x)v(x+1) + u(x)v(x+1) - u(x)v(x)$. Two terms cancel, leaving $u(x+1)v(x+1) - u(x)v(x) = \Delta(uv)$.

3. Let $n \in \mathbb{N}_0$. Calculate and simplify $\sum_0^n x^{\underline{1}} x^{\underline{10}} \delta x$.

Warning: $x^{\underline{1}} x^{\underline{10}} \neq x^{\underline{11}}$.

Method 1: Write $x^{\underline{1}} = x = (x - 10 + 10)$. Hence, $\sum_0^n x^{\underline{1}} x^{\underline{10}} \delta x = \sum_0^n (x - 10 + 10) x^{\underline{10}} \delta x = \sum_0^n (x - 10)^{\underline{1}} x^{\underline{10}} \delta x + \sum_0^n 10 x^{\underline{10}} \delta x = \sum_0^n x^{\underline{11}} \delta x + 10 \sum_0^n x^{\underline{10}} \delta x = \frac{1}{12} x^{\underline{12}} + \frac{10}{11} x^{\underline{11}}|_0^n = \frac{1}{132}(11n^{\underline{12}} + 120n^{\underline{11}} - (0 - 0)) = \frac{1}{132}(11n^{\underline{11}}(n-11)^{\underline{1}} + 120n^{\underline{11}}) = \frac{n^{\underline{11}}}{132}(11(n-11) + 120) = \frac{n^{\underline{11}}}{132}(11n - 1)$.

Method 2: Summation by parts. Set $u = x^{\underline{1}}, \Delta v = x^{\underline{10}}$. We have $\Delta u = x^{\underline{0}}, v = \frac{1}{11} x^{\underline{11}}$. Now, $\sum_0^n x^{\underline{1}} x^{\underline{10}} \delta x = x^{\underline{1}} \frac{1}{11} x^{\underline{11}}|_0^n - \sum_0^n \frac{1}{11}(x+1)^{\underline{11}} x^{\underline{0}} \delta x = (\frac{n \cdot n^{\underline{11}}}{11} - 0) - \frac{1}{11}\sum_0^n (x+1)^{\underline{11}} \delta x$. We reindex, setting $y = x + 1$, getting $\frac{n \cdot n^{\underline{11}}}{11} - \frac{1}{11} \sum_1^{n+1} y^{\underline{11}} \delta y = \frac{n \cdot n^{\underline{11}}}{11} - \frac{1}{11}\frac{1}{12} y^{\underline{12}}|_1^{n+1} = \frac{n \cdot n^{\underline{11}}}{11} - (\frac{(n+1)^{\underline{12}}}{132} - \frac{0}{132}) = \frac{n^{\underline{11}}}{132}(12n - (n+1)) = \frac{n^{\underline{11}}}{132}(11n - 1)$.

4. Let f, g be functions from $\mathbb{Z}$ to $\mathbb{R}$. Suppose that $\Delta f = \Delta g$. Prove that there is some constant C such that $f(x) = g(x) + C$.

Lemma: If $\Delta h = 0$, then there is some constant C with $h(x) = C$.
Proof: Set $C = h(0)$. We prove $\forall n \in \mathbb{N}_0$, $h(n) = C$ by induction. Base case: $h(0) = C$ already. Now, assume that $h(n) = C$. We have $0 = \Delta h = h(n+1) - h(n)$, so $h(n+1) = h(n) = C$. A similar proof works for all negative integer n.

Now, set $h(x) = f(x) - g(x)$. We have $\Delta h = f(x+1) - g(x+1) - (f(x) - g(x)) = (f(x+1) - f(x)) - (g(x+1) - g(x)) = \Delta f - \Delta g = 0 - 0 = 0$. Hence, by lemma, there is some constant C with $h(x) = C$. So, $f(x) - g(x) = C$, which rearranges to $f(x) = g(x) + C$.

5. Let $n \in \mathbb{N}$. Calculate $\sum_1^n H_x \delta x$.

We rewrite $\sum_1^n H_x \delta x = \sum_1^n x^{\underline{0}} H_x \delta x$, and use summation by parts. Set $u = H_x, \Delta v = x^{\underline{0}}$. We have $\Delta u = x^{\underline{-1}}$ and $v = x^{\underline{1}}$. Hence, $\sum_1^n x^{\underline{0}} H_x \delta x = x^{\underline{1}} H_x|_1^n - \sum_1^n (x+1)^{\underline{1}} x^{\underline{-1}} \delta x = (nH_n - 1H_1) - \sum_1^n x^{\underline{0}} \delta x = nH_n - 1 - x^{\underline{1}}|_1^n = nH_n - 1 - (n-1) = nH_n - n$.

6. Recall that $x^{\overline{m}} = \begin{cases} x(x+1)\cdots(x+m-1) & m \geq 0 \\ \frac{1}{(x-1)(x-2)\cdots(x+m)} & m \leq 0 \end{cases}$. Define the "other" difference operator Δ' as $\Delta' f = f(x) - f(x-1)$. Compute and simplify $\Delta' x^{\overline{m}}$.

For $m \geq 1$, we have $\Delta' x^{\overline{m}} = x(x+1)\cdots(x+m-2)(x+m-1) - (x-1)(x)\cdots(x+m-2) = x(x+1)\cdots(x+m-2)[x+m-1-(x-1)] = mx^{\overline{m-1}}$.

For $m \leq 0$, we have $\Delta' x^{\overline{m}} = \frac{1}{(x-1)(x-2)\cdots(x+m)} - \frac{1}{(x-2)(x-3)\cdots(x+m)(x+m-1)} = \frac{x+m-1}{(x-1)(x-2)\cdots(x+m)(x+m-1)} - \frac{x-1}{(x-1)(x-2)\cdots(x+m)(x+m-1)} = \frac{m}{(x-1)(x-2)\cdots(x+m)(x+m-1)} = mx^{\overline{m-1}}$.

Hence, in both cases, the result is $\Delta' x^{\overline{m}} = mx^{\overline{m-1}}$.