MATH 579: Combinatorics

Homework 7 Solutions

1. $a_0 = a_1 = 2,\ a_n = -2a_{n-1} - a_{n-2}\ (n \geq 2)$

   The characteristic equation is $x^2 = -2x - 1$, which has a double root of $r = -1$. Hence the general solution is $a_n = \alpha_1(-1)^n + \alpha_2 n(-1)^n$. Our initial conditions give $2 = a_0 = \alpha_1(-1)^0 + \alpha_2 0(-1)^0 = A$ and $2 = a_1 = \alpha_1(-1)^1 + \alpha_2 1(-1)^1 = -\alpha_1 - \alpha_2$. This has solution $\alpha_1 = 2, \alpha_2 = -4$, so our specific solution is $a_n = 2(-1)^n - 4n(-1)^n = 2(-1)^n(1-2n)$.

2. $a_0 = 0, a_1 = 1,\ a_n = 4a_{n-2}\ (n \geq 2)$

   The characteristic equation is $x^2 = 4$, which has roots $r_1 = 2, r_2 = -2$. Hence the general solution is $a_n = \alpha_1 2^n + \alpha_2(-2)^n$. Our initial conditions give $0 = a_0 = \alpha_1 2^0 + \alpha_2(-2)^0 = \alpha_1 + \alpha_2$ and $1 = a_1 = \alpha_1 2^1 + \alpha_2(-2)^1 = 2\alpha_1 - 2\alpha_2$. This has solution $\alpha_1 = 0.25,\ \alpha_2 = -0.25$, so our specific solution is $a_n = (0.25)2^n + (-0.25)(-2)^n = 2^{-2}2^n - (-2)^{-2}(-2)^n = 2^{n-2} - (-2)^{n-2}$.

3. $a_0 = 2, a_1 = -4, a_2 = 26, a_n = a_{n-1} + 8a_{n-2} - 12a_{n-3}\ (n \geq 3)$

   The characteristic equation is $x^3 = x^2 + 8x - 12$, which has $r_1 = -3$ as a single root, and $r_2 = 2$ as a double root. Hence the general solution is $a_n = \alpha_1(-3)^n + \alpha_2 2^n + \alpha_3 n2^n$. Our initial conditions give $2 = a_0 = \alpha_1(-3)^0 + \alpha_2 2^0 + \alpha_3 0 2^0 = \alpha_1 + \alpha_2$, $-4 = a_1 = \alpha_1(-3)^1 + \alpha_2 2^1 + \alpha_3 1 \cdot 2^1 = -3\alpha_1 + 2\alpha_2 + 2\alpha_3$, and $26 = a_2 = \alpha_1(-3)^2 + \alpha_2 2^2 + \alpha_3 2 \cdot 2^2 = 9\alpha_1 + 4\alpha_2 + 8\alpha_3$. This has solution $\alpha_1 = 2, \alpha_2 = 0, \alpha_3 = 1$, so our specific solution is $a_n = 2(-3)^n + n2^n$.

4. $a_0 = 0, a_1 = 0, a_2 = 0, a_n = 9a_{n-1} - 27a_{n-2} + 27a_{n-3}\ (n \geq 3)$

   The characteristic equation is $x^3 = 9x^2 - 27x + 27$, which has $r = 3$ as a triple root. Hence the general solution is $a_n = \alpha_1 3^n + \alpha_2 n3^n + \alpha_3 n^2 3^n$. Our initial conditions give $0 = a_0 = \alpha_1 3^0 + \alpha_2 0 \cdot 3^0 + \alpha_3 0^2 3^0 = \alpha_1$, $0 = a_1 = \alpha_1 3^1 + \alpha_2 1 \cdot 3^1 + \alpha_3 1^2 3^1 = 3\alpha_1 + 3\alpha_2 + 3\alpha_3$, and $0 = a_2 = \alpha_1 3^2 + \alpha_2 2 \cdot 3^2 + \alpha_3 2^2 3^2 = 9\alpha_1 + 18\alpha_2 + 36\alpha_3$. This has solution $\alpha_1 = \alpha_2 = \alpha_3 = 0$, so our specific solution is $a_n = 0$.

5. $a_0 = a_1 = 0, a_n = a_{n-1} + 2a_{n-2} + 3\ (n \geq 2)$

   We start with homogeneous equation $a_n = a_{n-1} + 2a_{n-2}$, which has characteristic equation $x^2 = x + 2$ and roots $r_1 = -1, r_2 = 2$. Hence the general solution is $a_n = \alpha_1(-1)^n + \alpha_2 2^n$. We now guess a solution to the nonhomogeneous recurrence, a zeroth-degree polynomial (constant) β. If $a_n = \beta$ solves the recurrence, then $\beta = \beta + 2\beta + 3$. This has unique solution $\beta = -1.5$, so our general solution to the nonhomogeneous recurrence is $a_n = \alpha_1(-1)^n + \alpha_2 2^n - 1.5$. Our initial conditions give $0 = a_0 = \alpha_1(-1)^0 + \alpha_2 2^0 - 1.5 = \alpha_1 + \alpha_2 - 1.5$ and $0 = a_1 = \alpha_1(-1)^1 + \alpha_2 2^1 - 1.5 = -\alpha_1 + 2\alpha_2 - 1.5$. This has solution $\alpha_1 = 0.5, \alpha_2 = 1$, so our specific solution is $a_n = 0.5(-1)^n + 2^n - 1.5 = \frac{(-1)^n + 2^{n+1} - 3}{2}$.

6. $a_0 = a_1 = 0, a_n = a_{n-1} + 2a_{n-2} + n\ (n \geq 2)$

   The homogeneous equation is the same as Problem 6, with general solution $a_n = \alpha_1(-1)^n + \alpha_2 2^n$. Now we guess a first-degree polynomial solution $\beta n + \gamma$ to the nonhomogeneous recurrence. Plugging in, we get $\beta n + \gamma = (\beta(n-1) + \gamma) + 2(\beta(n-2) + \gamma) + n = (3\beta + 1)n + (-5\beta + 3\gamma)$. Hence $\beta = 3\beta + 1$ and $\gamma = -5\beta + 3\gamma$, which has solution $\beta = -0.5, \gamma = -1.25$. Thus our general solution to the nonhomogeneous recurrence is $a_n = \alpha_1(-1)^n + \alpha_2 2^n - 0.5n - 1.25$. Our initial conditions give $0 = a_0 = \alpha_1(-1)^0 + \alpha_2 2^0 - 0.5 \cdot 0 - 1.25 = \alpha_1 + \alpha_2 - 1.25$ and $0 = a_1 = \alpha_1(-1)^1 + \alpha_2 2^1 - 0.5 \cdot 1 - 1.25 = -\alpha_1 + 2\alpha_2 - 1.75$. This has solution $\alpha_1 = \frac{1}{4}$, $\alpha_2 = 1$. Hence our specific solution is $a_n = \frac{1}{4}(-1)^n + 2^n - 0.5n - 1.25 = \frac{(-1)^n + 2^{n+2} - 2n - 5}{4}$.

7. $a_0 = a_1 = 0, a_n = a_{n-1} + 2a_{n-2} + e^n\ (n \geq 2)$

   The homogeneous equation is the same as Problem 6, with general solution $a_n = \alpha_1(-1)^n + \alpha_2 2^n$. Now we guess a solution βe^n to the nonhomogeneous recurrence. Plugging in, we get $\beta e^n = \beta e^{n-1} + 2\beta e^{n-2} + e^n$. Cancelling e^{n-2}, we get $\beta e^2 = \beta e + 2\beta + e^2$. This has unique solution $\beta = \frac{e^2}{e^2 - e - 2} \approx 2.77$. Thus our general solution to the nonhomogeneous recurrence is $a_n = \alpha_1(-1)^n + \alpha_2 2^n + \beta e^n$. Our initial conditions give $0 = a_0 = \alpha_1(-1)^0 + \alpha_2 2^0 + \beta e^0 = \alpha_1 + \alpha_2 + \beta$ and $0 = a_1 = \alpha_1(-1)^1 + \alpha_2 2^1 + \beta e^1 = -\alpha_1 + 2\alpha_2 + \beta e$. This has solution $\alpha_1 = \frac{e^2}{3+3e} \approx 0.66, \alpha_2 = \frac{e^2}{6-3e} \approx -3.43$. Hence our specific solution is $a_n = \frac{(-1)^n e^2}{3+3e} + \frac{2^n e^2}{6-3e} + \frac{e^{n+2}}{e^2 - e - 2}$.

8. What is the maximum number of regions we can divide the plane into, using n lines?

   We define the first few terms of sequence a_n via $a_0 = 1, a_1 = 2, a_2 = 4$. Suppose now we have $n-1$ lines already placed, and add the next. If it crosses none of the existing lines (e.g. all are parallel), we add one new region. If it crosses one of the existing lines, it adds one new region before the crossing, and one after the crossing. Since the new line can cross at most $n-1$ lines, it can add at most $(n-1)+1=n$ new regions, so $a_n = a_{n-1}+n$. The homogeneous equation has characteristic equation $x=1$, with general solution $a_n=\alpha$. We guess a second-degree polynomial solution $\tau n^2+\beta n+\gamma$ to the nonhomogeneous recurrence (the homogeneous solution overlaps, and no first-degree polynomial works). Plugging in, we get $\tau n^2+\beta n+\gamma = \tau(n-1)^2+\beta(n-1)+\gamma+n = \tau n^2+(-2\tau+\beta+1)n+(\tau-\beta+\gamma)$. It turns out γ can be anything (since it's absorbed into α), and $\tau=\beta=0.5$. Hence the general solution is $a_n=\alpha+\frac{n^2+n}{2}$. Our initial condition gives $1=a_0=\alpha+\frac{0^2+0}{2}$, so $\alpha=1$ and our specific solution is $a_n=1+\frac{n^2+n}{2}=\frac{n^2+n+2}{2}$.

9. Let a_n be the number of n-digit nonnegative integers in which no three consecutive digits are the same. Justify that $a_{n+2}=9a_{n+1}+9a_n$, then find a_n.

Call integers that satisfy this condition 'valid'. We divide valid integers with $n+2$ digits into two types: (A) those whose last two digits are the same, and (B) those whose last two digits are different. To count type (A), we take any valid integer with n digits, and append one of $00, 11, \ldots, 99$. However, one of these ten is forbidden (else we would have an invalid integer). Hence there are $9a_n$ of type (A). To count type (B), we take any valid integer with $n+1$ digits, and append one of $0, 1, \ldots, 9$. However, one of these ten is forbidden (else we wouldn't be of type B). Hence there are $9a_{n+1}$ of type (B). Adding, we get the desired recurrence relation. Note that it's only valid for $n\geq 1$, so we need to compute the two initial conditions $a_1=9, a_2=90$ separately. Our characteristic equation is $x^2=9x+9$, with roots $r_1=\frac{9+3\sqrt{13}}{2}, r_2=\frac{9-3\sqrt{13}}{2}$. The general solution is $a_n=\alpha_1 r_1^n+\alpha_2 r_2^n$. Applying our initial conditions, we get $9=a_1=\alpha_1 r_1+\alpha_2 r_2$ and $90=a_2=\alpha_1 r_1^2+\alpha_2 r_2^2$. These scary equations have solution $\alpha_1=\frac{1}{2}+\frac{3}{2\sqrt{13}}, \alpha_2=\frac{1}{2}-\frac{3}{2\sqrt{13}}$. Hence the specific solution is just $a_n=(\frac{1}{2}+\frac{3}{2\sqrt{13}})r_1^n+(\frac{1}{2}-\frac{3}{2\sqrt{13}})r_2^n$.

10. Let a_n be the number of ways to color the squares of a $1\times n$ chessboard using the colors red, white, and blue, so that no two red squares are adjacent.

Call colorings that satisfy this condition 'valid'. We divide valid colorings with $n+2$ squares into two types: (A) those whose last square isn't red, and (B) those whose last square is red. To count type (A), we take any valid coloring with $n+1$ squares, and color the last square white or blue. Hence there are $2a_{n+1}$ of type (A). To count type (B), the last square is red, so the next-to-last square must be white or blue, and the remaining squares are a valid coloring with n squares. Hence there are $2a_n$ of type (B). Combining, we get $a_{n+2}=2a_{n+1}+2a_n$. This is valid for $n\geq 1$, so we need to compute the initial conditions $a_1=3, a_2=8$ separately. Our characteristic equation is $x^2=2x+2$, with roots $r_1=1+\sqrt{3}$ and $r_2=1-\sqrt{3}$. Our general solution is $a_n=\alpha_1(1+\sqrt{3})^n+\alpha_2(1-\sqrt{3})^n$. Applying our initial conditions, we get $3=a_1=\alpha_1(1+\sqrt{3})+\alpha_2(1-\sqrt{3})$ and $8=\alpha_1(1+\sqrt{3})^2+\alpha_2(1-\sqrt{3})^2$. This has solution $\alpha_1=\frac{1}{2}+\frac{1}{\sqrt{3}}, \alpha_2=\frac{1}{2}-\frac{1}{\sqrt{3}}$. Hence our specific solution is $a_n=(\frac{1}{2}+\frac{1}{\sqrt{3}})(1+\sqrt{3})^n+(\frac{1}{2}-\frac{1}{\sqrt{3}})(1-\sqrt{3})^n$.

11. Let a_n be the number of ways to color the squares of a $1\times n$ chessboard using the colors red, white, and blue, so that no red square is adjacent to a white square. Justify the relation $a_{n+2}=2a_{n+1}+a_n$ (for certain n), and then find a_n.

Call colorings that satisfy this condition 'valid'. We divide valid colorings with $n+2$ squares into two types: (A) those whose next-to-last square is blue, and (B) those whose next-to-last square isn't blue. To count type (A), we take any valid coloring of the first n squares, and any coloring of the last square. This gives $3a_n$ of type (A). To count (B), we note that there are a_n valid colorings of $n+1$ squares, whose last square *is* blue, and a_{n+1} valid colorings of $n+1$ squares altogether. Hence there are $a_{n+1}-a_n$ valid colorings of $n+1$ squares, whose last square *isn't* blue. For each of these, there are two ways of coloring the last square – either blue, or repeating the next-to-last square. Hence there are $2(a_{n+1}-a_n)$ valid colorings of type (B). Combining, we get $a_{n+2}=2(a_{n+1}-a_n)+3a_n=2a_{n+1}+a_n$, valid for $n\geq 1$. We also compute initial conditions $a_1=3, a_2=7$. We have characteristic equation $x^2=2x+1$, with roots $r_1=1+\sqrt{2}, r_2=1-\sqrt{2}$. Hence the general solution is $a_n=\alpha_1(1+\sqrt{2})^n+\alpha_2(1-\sqrt{2})^n$. Applying our initial conditions, we get $3=a_1=\alpha_1(1+\sqrt{2})^1+\alpha_2(1-\sqrt{2})^1$ and $7=a_2=\alpha_1(1+\sqrt{2})^2+\alpha_2(1-\sqrt{2})^2$. This has solutions $\alpha_1=\frac{1+\sqrt{2}}{2}, \alpha_2=\frac{1-\sqrt{2}}{2}$. Hence our solution is $a_n=\frac{(1+\sqrt{2})^{n+1}+(1-\sqrt{2})^{n+1}}{2}$.

12. Let a_n be the number of ways to color the squares of a $1\times n$ chessboard using the colors red, white, and blue, so that the specific sequence red-white-blue does not occur. Find a recurrence that this sequence satisfies.

Call colorings that satisfy this condition 'valid'; we count valid colorings of length $n+3$. We might start with a valid coloring of length $n+2$, and then append any of 3 colors. This gives $3a_{n+2}$, but unfortunately some invalid colorings are included among these. The only way the coloring could be invalid is if the last three colors are exactly red-white-blue (no earlier red-white-blue could have occurred since we started with a valid coloring before appending blue). How many of these invalid ones snuck in? Exactly a_n of them – every valid coloring of length n, to be followed by red-white in our valid coloring of length $n+2$. Hence the sequence satisfies the recurrence relation $a_{n+3}=3a_{n+2}-a_n$, for all $n\geq 1$. We also need initial conditions $a_1=3, a_2=9, a_3=26$. We could finish this problem and solve the recurrence using our methods, except that the roots are really ugly.