MATH 579: Combinatorics
Homework 2 Solutions

1. Calculate $S(5,3)$ in two ways: with the formula involving binomial coefficients, and with the recurrence relation (and boundary conditions).

   First method: $S(5,3) = \frac{1}{3!}\sum_{j=0}^{3}(-1)^{3-j}\binom{3}{j}j^5 = \frac{1}{6}[-\binom{3}{0}0^5 + \binom{3}{1}1^5 - \binom{3}{2}2^5 + \binom{3}{3}3^5] = \frac{1}{6}(-0+3-96+243) = \frac{150}{6} = 25$.

   Second method: Since $S(n,1) = S(n,n) = 1$ for all $n \in \mathbb{N}$, we will just calculate the non-boundary values we will need: $S(3,2) = 2S(2,2) + S(2,1) = 3$. $S(4,2) = 2S(3,2) + S(3,1) = 7$. $S(4,3) = 3S(3,3) + S(3,2) = 6$. Finally, $S(5,3) = 3S(4,3) + S(4,2) = 25$.

2. Explicitly find all partitions of $\{a,b,c,d,e\}$ into three nonempty parts.

   We should have $S(5,3) = 25$ of these. It is very helpful to write in some nice order, to not repeat. $abc \cdot d \cdot e, abd \cdot c \cdot e, abe \cdot c \cdot d, acd \cdot b \cdot e, ace \cdot b \cdot d, ade \cdot b \cdot c, bcd \cdot a \cdot e, bce \cdot a \cdot d, bde \cdot a \cdot c, cde \cdot a \cdot b, ab \cdot cd \cdot e, ab \cdot ce \cdot d, ab \cdot de \cdot c, ac \cdot bd \cdot e, ac \cdot be \cdot d, ac \cdot de \cdot b, ad \cdot bc \cdot e, ad \cdot be \cdot c, ad \cdot ce \cdot b, ae \cdot bc \cdot d, ae \cdot bd \cdot c, ae \cdot cd \cdot b, a \cdot bc \cdot de, a \cdot bd \cdot ce, a \cdot be \cdot cd$.

3. Explicitly find all lists of length four, drawn from $[3]$, using each of $1,2,3$ at least once.

   We should have $3!S(4,3) = 36$ of these. Fortunately, there's a nice symmetry to exploit, as just one element is repeated in each list. There should be twelve that repeat 1. In lexicographic order, they are: $(1,1,2,3), (1,1,3,2), (1,2,1,3), (1,2,3,1), (1,3,1,2), (1,3,2,1), (2,1,1,3), (2,1,3,1), (2,3,1,1), (3,1,1,2), (3,1,2,1), (3,2,1,1)$. We can now take these and swap 1 with 2, to get the twelve that repeat 2: $(2,2,1,3), (2,2,3,1), (2,1,2,3), (2,1,3,2), (2,3,2,1), (2,3,1,2), (1,2,2,3), (1,2,3,2), (1,3,2,2), (3,2,2,1), (3,2,1,2), (3,1,2,2)$. We can also take our first twelve, and swap 1 with 3, to get the twelve that repeat 3: $(3,3,2,1), (3,3,1,2), (3,2,3,1), (3,2,1,3), (3,1,3,2), (3,1,2,3), (2,3,3,1), (2,3,1,3), (2,1,3,3), (1,3,3,2), (1,3,2,3), (1,2,3,3)$. Of course, we could have just calculated all 36 in lexicographic order.

4. Explicitly find all partitions of $\{a,b,c,d\}$ into any number of parts.

   There should be $B_4 = 15$ of these, corresponding to $S(4,1) + S(4,2) + S(4,3) + S(4,4)$. They are $abcd, abc \cdot d, abd \cdot c, acd \cdot b, bcd \cdot a, ab \cdot cd, ac \cdot bd, ad \cdot bc, ab \cdot c \cdot d, ac \cdot b \cdot d, ad \cdot b \cdot c, bc \cdot a \cdot d, bd \cdot a \cdot c, cd \cdot a \cdot b, a \cdot b \cdot c \cdot d$.

5. Explicitly find all lists of length three, drawn from $[n]$ for some $n \in \mathbb{N}$, using each of $1,2,\ldots,n$ at least once.

   There should be $1!S(3,1) + 2!S(3,2) + 3!S(3,3) = 13$ of these. For $n = 1$ there is just $(1,1,1)$. For $n = 2$ there are $(1,1,2), (1,2,1), (2,1,1), (2,2,1), (2,1,2), (1,2,2)$. For $n = 3$ there are $(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1)$. For $n \geq 4$ there are no such lists.

6. Determine the number of factorizations of 2310 into integers greater than 1. For example, 2310 and $2 \cdot 1155$ are two of these.

   We factor $2310 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$. Each factorization is a partition of the set $\{2,3,5,7,11\}$. Hence there are $B_5 = 52$ of them.

7. Prove the boundary conditions $S(n,1) = S(n,n) = 1$, for all $n \in \mathbb{N}$.

   The only partition of $[n]$ into one part is to put everything into that part. The only partition of $[n]$ into n parts is to put each element into its own part.

8. Prove the recurrence relation $S(n+1,k) = kS(n,k) + S(n,k-1)$, for $n \geq k \geq 1$.

We count partitions of $[n+1]$ into k parts in two ways. One way is just with $S(n+1,k)$. The other way is by considering $n+1$ separately. If it is in a part by itself, then the remaining $k-1$ parts form a partition of $[n]$. There are $S(n,k-1)$ such partitions. If instead $n+1$ is not alone, then removing it leaves a partition of $[n]$ into k parts. There are $S(n,k)$ such partitions, and k choices of which part to attach $n+1$ to. Hence there are $kS(n,k)$ partitions of $[n+1]$ into k parts where $n+1$ is not lonely. Adding, we get the desired result.

9. Prove that there are $n!S(k,n)$ lists of size k, drawn from $[n]$, using each of $1,2,\ldots n$ at least once.

We need a bijection between lists and partitions. Start with a list of size k, $(a_1,a_2,\ldots,a_k)$, whose elements are drawn from $[n]$, using each of $1,2,\ldots,n$ at least once. We assume $k \geq n$ else the answer is 0.

We put 1 into part a_1, we put 2 into part a_2, $\ldots$, and we put k into part a_k. There are potentially n parts, since the a_i are drawn from $[n]$. In fact, there are exactly n parts, since each of $1,2,\ldots,n$ are used at least once in the list. So, we have a function from lists of our type, to partitions of $[k]$ into n parts.

Unfortunately our function is not a bijection, because multiple lists go to the same partition. The reason is that partitions do not order the n parts, while our function does. For example, $(1,1,2,2)$ maps to partition $12 \cdot 34$, and $(2,2,1,1)$ maps to partition $34 \cdot 12$. These are the same partition. Luckily, though, this funciton is exactly $n!$-to-one, as there are $n!$ orderings of the n parts, in any fixed partition into n parts.

Hence there are just $n!$ times as many lists of size k, drawn from $[n]$, using each of $1,2,\ldots,n$ at least once, as there are partitions of $[k]$ into n parts, which is known to be $S(k,n)$.

10. Prove that $x^n = \sum_{k=1}^n S(n,k)x^{\underline{k}}$, for all $n \in \mathbb{N}$.

Base case: For $n=1$, the statement is $x^1 = S(1,1)x^{\underline{1}}$, which is true since $S(1,1) = 1$ and $x^{\underline{1}} = x^1 = x$.

Inductive case: Assume that $x^n = \sum_{k=1}^n S(n,k)x^{\underline{k}}$ holds. We multiply both sides by x to get $x^{n+1} = \sum_{k=1}^n S(n,k)x \cdot x^{\underline{k}}$. We now expand $x \cdot x^{\underline{k}} = x \cdot x \cdot (x-1) \cdot (x-2) \cdots (x-k+1) = (x-k+k) \cdot x \cdot (x-1) \cdot (x-2) \cdots (x-k+1) = x^{\underline{k+1}} + kx^{\underline{k}}$. Substituting this back, we get $x^{n+1} = \sum_{k=1}^n S(n,k)[x^{\underline{k+1}} + kx^{\underline{k}}] = \sum_{k=1}^n S(n,k)x^{\underline{k+1}} + \sum_{k=1}^n S(n,k)kx^{\underline{k}} =^1 \sum_{j=2}^{n+1} S(n,j-1)x^{\underline{j}} + \sum_{k=1}^n S(n,k)kx^{\underline{k}} =^2 \sum_{j=1}^{n+1} S(n,j-1)x^{\underline{j}} + \sum_{k=1}^{n+1} S(n,k)kx^{\underline{k}} =^3 \sum_{k=1}^{n+1}[S(n,k-1) + kS(n,k)]x^{\underline{k}} =^4 \sum_{k=1}^{n+1} S(n+1,k)x^{\underline{k}}$.

[1]: Reindexing via $j = k+1$
[2]: Since $S(n,0) = 0 = S(n,n+1)$
[3]: Renaming j back to k so we can combine sums.
[4]: Using the recurrence relation.