MATH 579: Combinatorics

Homework 1 Solutions

For these next problems, a "word" is a string of letters, drawn from the 26 options ab...z. n represents an arbitrary natural number; solve the problems for all n.

1. How many words are there of length n?
   These are equivalent to lists of length n, drawn from $[26]$. Our answer is 26^n.

2. How many words are there of length n, with all different letters?
   If $n > 26$, then the answer is 0. Otherwise the answer is $26^{\underline{n}}$. Note that our definition of falling powers does not define expressions like $26^{\underline{30}}$. This problem makes a good case that these should be defined to be zero.

3. How many words are there of length n, using each of the 26 letters at least once?
   If $n < 26$ then the answer is 0. Otherwise the answer is $26!S(n, 26)$.

4. How many words are there of length n, with no vowels?
   The restriction reduces our 26 options to 21; our problem is equivalent to repeating problem 1 but drawing from $[21]$ instead of $[26]$. Our answer is 21^n.

5. How many words are there of length n, with at least one vowel and at least one consonant?
   By the previous problem, 21^n words of length n have no vowels. By a similar calculation, 5^n words of length n have no consonants. All words of length n either fall in one of the preceding, disjoint, categories, or else are of the type we want to count. Hence, $21^n + 5^n + x = 26^n$. We subtract, to find our desired $x = 26^n - 21^n - 5^n$. This is valid for all $n \in \mathbb{N}$.

6. How many words are there of length n, with the first three letters vowels, and the remaining letters consonants?
   There are 5^3 three-letter words consisting entirely of vowels. There are 21^{n-3} words of length $n-3$, consisting entirely of consonants. We select one of each; hence the answer is $5^3 21^{n-3}$. Note that this is only valid for $n \geq 3$. For $n = 1, 2$, the answer is 0.

7. How many words are there of length n, with exactly three a's?
   If $n = 1, 2$, the answer is 0. Now assume $n \geq 3$. We mark n places where we will place letters. Three of them will be a's; there are $\binom{n}{3}$ ways to pick those three locations. Think of a set of size 3 drawn from $[n]$, where the set marks the locations where a's will go.

   The remaining $n - 3$ spaces are filled consecutively with any word, drawn from the alphabet that contains no a (equivalent to $[25]$). There are 25^{n-3} ways to do this. Combining, our solution is $\binom{n}{3}25^{n-3}$.

8. How many words are there of length n, with exactly three a's, appearing consecutively?
   If $n = 1, 2$ the answer is 0; we assume $n \geq 3$. We again mark n places for letters, and place our three a's. There are $n - 2$ choices for the first a (it can't be in the last two spaces because then there wouldn't be enough room for the other a's afterward); the remaining a's will follow right afterwards. Then, we fill the remaining apces with non-a letters as previously. Hence, the solution is $(n - 2)25^{n-3}$.

9. How many words are there of length n, with no two consecutive letters being the same?
   This one is easier to solve by working through one letter at a time. There are 26 choices for the first letter. The second letter can be anything but the first letter, so 25 choices. The third letter can be anything but the second letter, so again 25 choices. This continues, so our answer is $26 \cdot 25^{n-1}$.

10. How many words are there of length n, How many words are there of length n, with all letters different, and also forbidding the two-letter combinations $ab, bc, cd, \ldots, xy, yz, za$?

Messed up problem, ignore.

11. How many words are there of length n, whose first and last letters are the same, and also second and second-to-last letters are the same, and so on?

The answer is different, depending on whether n is even or odd. If even, there are just $n/2$ such pairs. We choose the first half of the word freely; there are $26^{n/2}$ ways to do this. The second half of the word is then completely determined, so our answer is $26^{n/2}$.

If n is odd, then there are $(n-1)/2$ such pairs, and a single lonely letter in the middle with no friend. We can then pick the first $(n-1)/2+1 = (n+1)/2$ letters freely, and the remaining letters are determined. Our answer is $26^{(n+1)/2}$.

If you're familiar with the ceiling function, one can combine the two halves as $26^{\lceil \frac{n}{2} \rceil}$.

For these next problems, we are shopping. The store has 26 items for sale, numbered 1 to 26. We can buy an item more than once, and the order in which we buy items does not matter.

12. How many ways are there of buying n items?

This is a multiset of n objects, drawn from [26]. The answer is $\binom{26+n-1}{n} = \binom{25+n}{n}$.

13. How many ways are there of buying n items, all numbered with primes?

The primes in this range are $2, 3, 5, 7, 11, 13, 17, 19, 23$, i.e. 9 of them. Hence we reduce from [26] to [9]; our answer is $\binom{8+n}{n}$.

14. How many ways are there of buying n items, all different?

We switch from a multiset to a set. Provided that $n \leq 26$, the answer is $\binom{26}{n}$. If $n > 26$, the answer is 0.

15. How many ways are there of buying n items, ensuring that we buy all 26 items at least once?

We have a formula for this, and it is $\binom{n-1}{25}$. This is valid provided $n \geq 26$; otherwise the answer is 0.

16. How many ways are there of buying n items, all different, ensuring that we do not buy two items with consecutive numbers.

Following the hint, suppose we order the items we buy $a_1 < a_2 < \cdots < a_n$. But, in fact, not only are the item numbers all different, they cannot even be consecutive. Hence $a_1 < a_2 - 1$ and $a_2 < a_3 - 1$ and $a_3 < a_4 - 1$, and so on. We assemble this as $1 \leq a_1 < a_2 - 1 < a_3 - 2 < \cdots < a_n - (n-1) \leq 26 - (n-1)$. Set $a'_1 = a_1$, $a'_2 = a_2 - 1$, $a'_3 = a_3 - 2$, $\ldots a'_n = a_n - (n-1)$. We can think of this problem as choosing $a'_1, \ldots, a'_n$, a subset from $[1, 2, \ldots, 26-(n-1)]$. There are $\binom{26-(n-1)}{n}$ ways to do this. Note that if $n = 13$, this is $\binom{14}{13} = 14$; however if $n \geq 14$, this is undefined (and our answer is 0).