MATH 579: Combinatorics

Exam 2 Solutions

1. Use difference calculus to compute $\sum_{i=1}^{100} i^4$.

We have $\sum_{i=1}^{100} i^4 = \sum_1^{101} x^4 \delta x = \sum_1^{101} x^{\underline{1}} + 7x^{\underline{2}} + 6x^{\underline{3}} + x^{\underline{4}}\delta x$, using our table for $S(n,k)$. We continue as $\frac{1}{2}x^{\underline{2}} + \frac{7}{3}x^{\underline{3}} + \frac{6}{4}x^{\underline{4}} + \frac{1}{5}x^{\underline{5}}|_1^{101} = \frac{1}{2}(101)^{\underline{2}} + \frac{7}{3}(101)^{\underline{3}} + \frac{6}{4}(101)^{\underline{4}} + \frac{1}{5}(101)^{\underline{5}} - 0 = 2,050,333,330$.

2. Let $n \in \mathbb{N}$. Prove that $n\binom{2n-1}{n-1} = \sum_{k=0}^n k\binom{n}{k}^2$.

By the committee/chair identity, $k\binom{n}{k} = n\binom{n-1}{k-1}$. By symmetry, $n\binom{n-1}{k-1} = n\binom{n-1}{n-k}$. Hence $\sum_{k=0}^n k\binom{n}{k}^2 = \sum_{k=0}^n n\binom{n-1}{n-k}\binom{n}{k} = n\sum_{k=0}^n \binom{n-1}{n-k}\binom{n}{k} = n\binom{2n-1}{n}$. The last step follows from the Chu-Vandermonde identity with $x = n, y = n-1, a = n$. By symmetry, $n\binom{2n-1}{n} = n\binom{2n-1}{n-1}$.

3. Let $n \in \mathbb{N}$. Prove that $n(n+1)2^{n-2} = \sum_{k=0}^n k^2\binom{n}{k}$.

We begin with the binomial theorem, stating $(x+1)^n = \sum_{k=0}^n \binom{n}{k}x^k$. Taking $\frac{d}{dx}$ of both sides, we get $n(x+1)^n = \sum_{k=0}^n k\binom{n}{k}x^{k-1}$. Multiplying both sides by x, we get $nx(x+1)^{n-1} = \sum_{k=0}^n k\binom{n}{k}x^k$. Taking $\frac{d}{dx}$ of both sides, we get $n[(x+1)^{n-1} + (n-1)x(x+1)^{n-2}] = \sum_{k=0}^n k^2\binom{n}{k}x^{k-1}$. Taking $x = 1$, we get $\sum_{k=0}^n k^2\binom{n}{k} = n(2^{n-1} + (n-1)2^{n-2}) = n(n+1)2^{n-2}$.

4. Let $k \in \mathbb{Z}$, $x \in \mathbb{R}$ with $x > k-2 \geq 0$. Prove that $\binom{x}{k}\binom{x+2}{k} \leq \binom{x+1}{k}^2$.

Since $k \geq 2$, $-2k \leq -k$, and hence $x^2+3x-kx-2k+2 \leq x^2+3x-kx-k+2$. We factor each side to get $(x-k+1)(x+2) \leq (x-k+2)(x+1)$. Since $x > k-2$, $x-k+2$ is positive (and so is $x+1$). Hence we can divide by the positive RHS to conclude $\frac{(x-k+1)(x+2)}{(x-k+2)(x+1)} \leq 1$. Now, we compute $\binom{x}{k}\binom{x+2}{k} = \frac{1}{k!k!}x^{\underline{k}}(x+2)^{\underline{k}} = \frac{1}{k!k!}(x+1)^{\underline{k}}(x+1)^{\underline{k}}\frac{(x-k+1)(x+2)}{(x-k+2)(x+1)} \leq \frac{1}{k!k!}(x+1)^{\underline{k}}(x+1)^{\underline{k}} = \binom{x+1}{k}^2$.

5. Let $n, k \in \mathbb{Z}$ with $n > 1$ and $k > 1$. Prove that $k^n < \binom{nk}{n}$.

We compute $\binom{nk}{n} = \frac{(nk)^{\underline{n}}}{n!} = \frac{(nk-0)(nk-1)(nk-2)\cdots(nk-(n-1))}{n!} = \frac{nk-0}{n-0}\frac{nk-1}{n-1}\frac{nk-2}{n-2}\cdots\frac{nk-(n-1)}{n-(n-1)}$, which we can write as $\prod_{i=0}^{n-1}\frac{nk-i}{n-i}$. Now, $k > 1$ so for $i \geq 1$ we have $ki > i$, which rearranges to $nk - i > nk - ki$. Since $i \leq n-1$, we divide by the positive $n-i$ to get $\frac{nk-i}{n-i} > k$. For $i = 0$, $\frac{nk-i}{n-i} = k$. Hence $\binom{nk}{n} \geq k^n$ for $n \geq 1$, and for $n > 1$ the inequality is strict.

6. Compute $\sum_{k=1}^n \frac{H_k}{(k+1)(k+2)}$.

We write $\sum_{k=1}^n \frac{H_k}{(k+1)(k+2)} = \sum_1^{n+1} H_x x^{\underline{-2}}\delta x$. We set $u = H_x, \Delta v = x^{\underline{-2}}$. This gives $\Delta u = x^{\underline{-1}}$ and $v = -x^{\underline{-1}}$. We sum by parts, getting $\sum H_x x^{\underline{-2}}\delta x = -x^{\underline{-1}}H_x - \sum -(x+1)^{\underline{-1}}x^{\underline{-1}}\delta x = -x^{\underline{-1}}H_x + \sum x^{\underline{-2}}\delta x = -x^{\underline{-1}}H_x - x^{\underline{-1}} = -x^{\underline{-1}}(H_x+1)$. We evaluate from 1 to $n+1$, getting $-(n+1)^{\underline{-1}}(H_{n+1}+1) + 1^{\underline{-1}}(H_1+1) = -\frac{H_{n+1}+1}{n+2} + 1$. Note that as $n \to \infty$, the fraction approaches 0, so the sum approaches 1.