Math 524 Exam 2 Solutions

All problems are for the vector space $\mathbb{R}_2[t]$, real polynomials of degree at most 2. We define $V = \{p(t) : p(1) = 0\}$, a subspace of $\mathbb{R}_2[t]$.

1. Let $A = \{a_1, a_2\}$ for $a_1 = t - 1, a_2 = t^2 - 1$. Let $B = \{b_1, b_2\}$ for $b_1 = t^2 + t - 2, b_2 = t^2 + 2t - 3$. Prove that A and B are each bases of V.

   We first prove that A, B are independent. $\alpha a_1 + \beta a_2 = \beta t^2 + \alpha t + (-\alpha - \beta)$; if this equals zero then $\alpha = \beta = 0$. $\alpha b_1 + \beta b_2 = (\alpha+\beta)t^2 + (\alpha+2\beta)t + (-2\alpha-3\beta)$; if this equals zero then $\alpha + \beta = 0 = \alpha + 2\beta$. The only solution is $\alpha = \beta = 0$. Because $V \neq \mathbb{R}_2[t]$, which has dimension 3, V has dimension at most 2. However, it has dimension at least 2 since A, B are in V, independent, and of cardinality 2. Hence A, B are bases.

2. Calculate $[3t^2 - 5t + 2]_A$.

   Since only a_2 has a t^2 term, and only a_1 has a t term, this is easy: $(-5, 3)^T$. Note: this means that $3t^2 - 5t + 2 = (-5)(t-1) + (3)(t^2 - 1)$.

3. Calculate P_{BA}.

   It is easier to first find $P_{AB} = ([b_1]_A [b_2]_A) = ((1,1)^T (2,1)^T) = \left(\begin{smallmatrix} 1 & 2 \\ 1 & 1 \end{smallmatrix}\right)$.
   We now calculate $P_{BA} = P_{AB}^{-1} = \left(\begin{smallmatrix} -1 & 2 \\ 1 & -1 \end{smallmatrix}\right)$.

4. Use the results of the previous two problems to calculate $[3t^2 - 5t + 2]_B$.

   $[3t^2 - 5t + 2]_B = P_{BA}[3t^2 - 5t + 2]_A = \left(\begin{smallmatrix} -1 & 2 \\ 1 & -1 \end{smallmatrix}\right)\left(\begin{smallmatrix} -5 \\ 3 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 11 \\ -8 \end{smallmatrix}\right)$.
   Note: this means that $3t^2 - 5t + 2 = (11)(t^2 + t - 2) + (-8)(t^2 + 2t - 3)$.

5. Let $W = \{at : a \in \mathbb{R}\}$. This is a subspace of $\mathbb{R}_2[t]$. Prove that $\mathbb{R}_2[t]$ is the internal direct sum of V and W.

   This is a consequence of Theorem 2.13. We have been told (and we believe, being trusting people) that V, W are subspaces of $\mathbb{R}_2[t]$. To complete the proof, we need to show two things:

   1. The dimension of V (already calculated to be 2), plus the (unknown) dimension of W, equals the dimension of $\mathbb{R}_2[t]$ (already known to be 3).
   2. $V \cap W = \{0\}$.

   We prove that W is one-dimensional (1) by observing that every polynomial in W is a scalar multiple of every other; hence an independent set can have only one vector in it. We next note that for $f(t) = at$, an element of W, $f(1) = a$. Hence for this to be in V we must have $a = 0$; in this case $f(t) = 0$ which is the zero polynomial (zero vector). This proves (2).