Math 522 Exam 5 Solutions

1. Prove or disprove: $29^{76} \equiv 76^{29} \pmod{35}$.

   Easy way: Statement holds if and only if $29^{76} \equiv 76^{29} \pmod{5}$ *AND* $29^{76} \equiv 76^{29} \pmod{7}$. *Note that, modulo 5,* $29 \equiv -1$ *and* $76 \equiv 1$. *Also, modulo 7,* $29 \equiv 1$ *and* $76 \equiv -1$. *The first equation therefore simplifies to* $(-1)^{76} \equiv (1)^{29} \pmod{5}$, *which is true since* $(-1)^{76} = 1$. *The second equation, however, simplifies to* $(1)^{76} \equiv (-1)^{29} \pmod{7}$, *which is false. Hence the statement does not hold.*
   Note: It was not necessary to check modulo 5, I did this for completeness.

   Hard way: Working modulo 35, we see that $29 \equiv -6$ *and* $76 \equiv 6$. *Further,* $29^{76} \equiv (-6)^{76} \equiv ((-6)^2)^{38} \equiv 6^{76}$. *On the other hand,* $76^{29} \equiv 6^{29}$. *Hence, the problem is equivalent to* $6^{76} \equiv 6^{29} \pmod{35}$. *Let's calculate powers of 6, modulo 35. Imediately we see that* $6^2 \equiv 1$. *Hence* $6^{28} \equiv (6^2)^{14} \equiv 1$. *But then* $6^{29} = 6^{28}6^1 \equiv 6 \not\equiv 1 \equiv (6^2)^{38} = 6^{76}$.

   Completely mechanical way: We first calculate powers of 29, modulo 35. We immediately see that $29^2 \equiv 1$, *and hence* $29^{76} = (29^2)^{37} \equiv 1$. *We now calculate powers of 76, modulo 35. We again find the extremely lucky situation that* $76^2 \equiv 1$, *and hence* $76^{29} \equiv (76^2)^{14}76^1 \equiv 76$. *It remains to check whether* $1 \equiv 76 \pmod{35}$, *which is false.*

2. Recall that $a \equiv b \pmod{c}$ means that $(a-b)/c$ is an integer. Recall also that $\lfloor \alpha \rfloor$ is the greatest integer less than or equal to α. For real numbers x, y ($y > 0$) define $f(x,y) = x - y\lfloor {}^{x}\!/_{y} \rfloor$.

   (a) Prove that $f(x,y) \equiv x \pmod{y}$.
   We calculate $(f(x,y) - x)/y = (x - y\lfloor {}^{x}\!/_{y} \rfloor - x)/y = -y\lfloor {}^{x}\!/_{y} \rfloor/y = -\lfloor {}^{x}\!/_{y} \rfloor$. *But this is an integer, by the definition of* $\lfloor \alpha \rfloor$.

   (b) Prove that $0 \le f(x,y) < y$
   The key fact is that $\alpha - 1 < \lfloor \alpha \rfloor \le \alpha$, *for any real number* α. *Because* $\lfloor \alpha \rfloor \le \alpha$, *we have* $-\lfloor \alpha \rfloor \ge -\alpha$. *Hence* $f(x,y) = x - y\lfloor {}^{x}\!/_{y} \rfloor \ge x - y{}^{x}\!/_{y} = x - x = 0$. *On the other hand,* $y - f(x,y) = y - x + y\lfloor {}^{x}\!/_{y} \rfloor > y - x + y({}^{x}\!/_{y} - 1) = y - x + x - y = 0$.

   Alternate solution: Use the similar fact, that $\alpha - \lfloor \alpha \rfloor = \beta$, *for some* $0 \le \beta < 1$. *Hence* $\lfloor {}^{x}\!/_{y} \rfloor = {}^{x}\!/_{y} - \beta$. *We substitute this into* $f(x,y) = x - y({}^{x}\!/_{y} - \beta) = y\beta$. *Since* $0 \le \beta < 1$ *and* $y > 0$, *the desired results follow.*

3. Exam grades: 100, 95, 94, 93, 87, 84, 81, 76, 72, 68, 65, 63, 52