Math 522 Exam 4 Solutions

1. Prove or disprove that $202|(3^{100{,}000}-1)$.

   The prime factorization of 202 *is* $2\cdot 101$. *We first show that* $101|(3^{100{,}000}-1)$. *Set* $m=3^{1{,}000}$. *By Fermat's little theorem,* $101|(m^{101}-m)=m(m^{100}-1)$. *Since* 101 *is prime, it must divide either* m *(which it doesn't, since* m *is a power of 3), or* $m^{100}-1$. *But* $m^{100}=(3^{1000})^{100}=3^{100{,}000}$, *so* $101|(3^{100{,}000}-1)$. *Hence* $3^{100{,}000}-1=101k$. *But* $3^{100{,}000}-1$ *is the difference of two odd numbers, hence is even. Since 101 is odd,* k *must be even, that is* $k=2m$. *So* $3^{100{,}000}-1=101\cdot 2\cdot m$.

2. The Lucas numbers L_i are defined as $L_0=2, L_1=1, L_{i+2}=L_{i+1}+L_i$ (for $i\in\mathbb{N}_0$). Find the generating function for the Lucas numbers.
   BONUS: Find a closed form for the Lucas numbers.

   Set $f(x)=L_0+L_1x+L_2x^2+L_3x^3+L_4x^4+\cdots$. *Then* $xf(x)=L_0x+L_1x^2+L_2x^3+L_3x^4+\cdots$ *and* $x^2f(x)=L_0x^2+L_1x^3+L_2x^4+\cdots$. *Adding, we get* $xf(x)+x^2f(x)=L_0x+(L_0+L_1)x^2+(L_1+L_2)x^3+(L_2+L_3)x^4+\cdots=2x+L_2x^2+L_3x^3+L_4x^4+\cdots$. *Add* $(2-x)$ *to both sides, to get* $2-x+xf(x)+x^2f(x)=2+x+L_2x^2+L_3x^3+L_4x^4+\cdots=f(x)$. *A bit of algebra gives* $2-x=f(x)(1-x-x^2)$ *and hence* $f(x)=(2-x)/(1-x-x^2)$.

   Several of you instead used $g(x)=L_0x+L_1x^2+L_2x^3+L_3x^4+L_4x^5+\cdots$. *This is not wrong, merely shifted (multiplied by* x*). In this case,* $g(x)=(2x-x^2)/(1-x-x^2)$.

   BONUS: $(2-x)/(1-x-x^2)=(x-2)/(x^2+x-1)=A/(x-\phi_1)+B/(x-\phi_2)$, *where (as with the Fibonacci numbers)* $\phi_1=-\frac{1+\sqrt{5}}{2}, \phi_2=-\frac{1-\sqrt{5}}{2}$. *We get the two equations* $A+B=1, -\phi_2A-\phi_1B=-2$. *This has solution* $A=-\phi_1, B=-\phi_2$. *We recall that* $1/(x-\gamma)=-\gamma^{-1}(1+\gamma^{-1}x+\gamma^{-2}x^2+\cdots)$; *hence* $f(x)=(1+\phi_1^{-1}x+\phi_1^{-2}x^2+\cdots)+(1+\phi_2^{-1}x+\phi_2^{-2}x^2+\cdots)$. *Combining terms, we see that* $L_i=\phi_1^{-i}+\phi_2^{-i}$.

3. Exam grades: 105, 101, 97, 93, 91, 90, 85, 85, 73, 73, 66, 62, 54