MATH 521B: Abstract Algebra
Homework 9: Due Apr. 13

1. Let G, H be groups. Prove that $G \times H \cong H \times G$.

2. Let G, H, K be groups. Prove that $(G \times H) \times K \cong G \times (H \times K)$. This associative property justifies writing just $G \times H \times K$.

3. Let G, H be finite groups. Prove that $|G \times H| = |G| \cdot |H|$.

4. Let G, H be groups. Prove that $G \times H$ is abelian if and only if both G and H are abelian.

5. Let $G = G_1 \oplus G_2$, an internal direct sum of finite abelian groups. Prove $G/G_1 \cong G_2$.

6. Let G_1, G_2 be finite groups, and let $a = (a_1, a_2) \in G_1 \times G_2$, an external direct product. Prove that $|a| = \text{lcm}(|a_1|, |a_2|)$.

7. Let $m, n \in \mathbb{N}$, with $\gcd(m, n) = 1$. Prove that $\mathbb{Z}_{mn} \cong \mathbb{Z}_m \times \mathbb{Z}_n$, an external direct product. This is called the Chinese Remainder Theorem.
(hint: build an internal direct sum.)

8. Let $m, n \in \mathbb{N}$, with $\gcd(m, n) \neq 1$. Prove that $\mathbb{Z}_m \times \mathbb{Z}_n$ is not cyclic.

9. Suppose $G_1 \trianglelefteq G$, $G_2 \trianglelefteq G$, and $G_1 \cap G_2 = \{id\}$. Prove that for every $g_1 \in G_1, g_2 \in G_2$, in fact $g_1 g_2 = g_2 g_1$. Note that this holds even if G, G_1, G_2 are each noncommutative.

10. Let $G = \mathbb{Z}_3 \times \mathbb{Z}_3$, and set $G_1 = \langle (0,1) \rangle, G_2 = \langle (1,0) \rangle, G_3 = \langle (1,1) \rangle$. Prove that $G = G_1 + G_2 + G_3$ and $\{(0,0)\} = G_1 \cap G_2 = G_1 \cap G_3 = G_2 \cap G_3$, and find some $g \in G$ with nonunique representation in $G_1 + G_2 + G_3$. This problem illustrates that it is not enough to check pairwise disjointness for internal direct sums; G is NOT an internal direct sum of G_1, G_2, G_3.

11. Let $G_1, G_2, \ldots$ be an infinite set of groups. We can define their direct product $\prod_i G_i$ as the set of all sequences $(a_1, a_2, \ldots)$ such that $a_i \in G_i$ for all i. We can imbue this with an operation in the natural way: $(a_1, a_2, \ldots)(b_1, b_2, \ldots) = (a_1 b_1, a_2 b_2, \ldots)$. Prove that this forms a group.

12. Let $G_1, G_2, \ldots$ be an infinite set of groups. We can define their direct sum $\sum_i G_i \subseteq \prod_i G_i$ as the set of all sequences $(a_1, a_2, \ldots)$ such that $a_i \in G_i$ for all i, with the added property that all but finitely many a_i are equal to the identity in G_i. Prove that $(\sum_i G_i) \trianglelefteq (\prod_i G_i)$.