MATH 521B: Abstract Algebra
Homework 7: Due Mar. 16

We now define a very important infinite non-abelian group, with operation matrix multiplication. The general linear group is $GL(2,\mathbb{R}) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : a,b,c,d \in \mathbb{R}, ad - bc \neq 0\}$.

1. Set $H = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : a,b,c,d \in \mathbb{R}, ad - bc > 0\}$. Prove that $H \trianglelefteq GL(2,\mathbb{R})$.
2. Set $K = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : a,b,c,d \in \mathbb{R}, ad - bc \in \mathbb{Q}, ad - bc \neq 0\}$. Prove that $K \trianglelefteq GL(2,\mathbb{R})$.
3. Set $M = \{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) : a,b,d \in \mathbb{R}, ad \neq 0\}$. Prove that $M \not\trianglelefteq GL(2,\mathbb{R})$.
4. Set $S = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : a,b,c,d \in \mathbb{R}, ad - bc = 1\}$. Prove that $S \trianglelefteq GL(2,\mathbb{R})$. This is called the special linear group and is denoted $SL(2,\mathbb{R})$.
5. Find the center Z of $SL(2,\mathbb{R})$. The quotient group $SL(2,\mathbb{R})/Z$ is called the projective special linear group and is denoted $PSL(2,\mathbb{R})$. It turns out to be isomorphic to the group of all complex Möbius transformations (aka linear fractional transformations) i.e. $f(z) = \frac{az+b}{cz+d}$.

We now turn to general groups G.

6. Suppose that $H \leq G$, and let $a \in G$. Prove that $aHa^{-1} \leq G$, and $|H| = |aHa^{-1}|$.
7. Suppose that $H \leq G$. Suppose that H is the only subgroup of G of order $|H|$. Prove that $H \trianglelefteq G$.
8. Suppose that $N \leq G$. Prove that $N \trianglelefteq G$ if and only if the product of any two right cosets of N is another right coset of N.
9. Suppose that $N \leq G$. Prove that $N \trianglelefteq G$ if and only if every left coset of N is a right coset of N.
10. Prove that $A_n \trianglelefteq S_n$, where S_n is the symmetric group and A_n is the alternating group (set of even permutations).

We recall the quaternion group Q, as defined in the previous homework.

11. Set $K = \{1, -1, i, -i\}$. Write down the multiplication table for the quotient group Q/K.
12. Set $H = \{1, -1\}$. Write down the multiplication table for the quotient group Q/H.
13. A group all of whose subgroups are normal is called a Dedekind group. Prove that Q is a non-abelian Dedekind group.