MATH 521B: Abstract Algebra
Homework 6: Due Mar. 9

We define the *quaternion group* Q. The elements are $Q = \{1, -1, i, -i, j, -j, k, -k\}$, and the non-commutative operation is multiplication. 1 is the identity as expected, $(-1)^2 = 1$, $i^2 = j^2 = k^2 = -1$, $ij = k$, $jk = i$, $ki = j$, $ji = -k$, $kj = -i$, $ik = -j$. (for problems 1,2)

1. Set $K = \{1, -1, i, -i\}$. Find all right cosets and left cosets of K in Q, and determine whether K is normal in Q.

2. Set $H = \{1, -1\}$. Find all right cosets and left cosets of H in Q, and determine whether H is normal in Q.

Let G be the set of isometries of the square (for problems 3-6).

3. Let K be the set of rotations of the square ($|K| = 4$). Find all right cosets and left cosets of K in G, and determine whether K is normal in G.

4. Let H be the set of isometries of the rectangle ($|H| = 4$). Find all right cosets and left cosets of H in G, and determine whether H is normal in G.

5. Let R be the subgroup generated by a reflection in a line through two corners ($|R| = 2$). Find all right cosets and left cosets of R in G, and determine whether R is normal in G.

6. Let F be the subgroup generated by the 180° rotation ($|F| = 2$). Find all right cosets and left cosets of F in G, and determine whether F is normal in G.

Let G be an arbitrary group (for problems 7-12).

7. Let K be a subgroup of G. Recall that $a \equiv b \pmod{K}$ means that $ab^{-1} \in K$. Prove that this is an equivalence relation, i.e. is reflexive, symmetric, and transitive.

8. Let K be a subgroup of G. Set $[a] = \{b \in G : a \equiv b \pmod{K}\}$, the equivalence class containing a. Prove that $[a] = Ka$.

9. Let K be a subgroup of G. Prove that the right cosets of K form a partition of G.

10. Let K be a subgroup of G, and $a \in G$. Without Lagrange's theorem, prove that $|Ka| = |K|$.

11. Let G be a group and $Z(G)$ be its center. Prove that $Z(G)$ is normal in G.

12. Let K be a subgroup of G. Suppose that G is abelian. Prove that K is normal in G.