MATH 521B: Abstract Algebra
Homework 4: Due Feb. 16

1. For each of the following five shapes in $\mathbb{R}^2$, the isometry group is isomorphic to a subgroup of the permutation group on the vertices. Draw the shape, label the vertices, and give a set of generators for the abovementioned subgroup.
   A: scalene triangle B: isosceles (not equilateral) triangle C: equilateral triangle
   D: rectangle (not square) E: square

2. We call a permutation a "cycle" if its cycle notation contains a single cycle (and all other elements are fixed). Prove that disjoint cycles commute.

3. Prove that every permutation may be written uniquely as a product of disjoint cycles. We call the number of each cycle length in this unique expression the "cycle structure" of the permutation. For example, (1 2)(3 4) has two cycles of length 2.

   For $a, b \in S_n$, we define $a^b = bab^{-1}$.

4. For $a = (1\ 2\ 3)(4\ 5)$, $b = (1\ 4)(3\ 5)$, calculate a^b and b^a.

5. Suppose $a, b \in S_n$ and $a = (a_1\ a_2\ \cdots\ a_k)$, a cycle. Prove that $a^b = (b(a_1)\ b(a_2)\ \cdots b(a_k))$.

6. Suppose $a, b \in S_n$. Prove that a and a^b have the same cycle structure.

7. Suppose $a, c \in S_n$. Suppose further that a, c have the same cycle structure. Prove that there is some $b \in S_n$ such that $c = a^b$.

8. Let $a, c \in S^n$. If there is some $b \in S_n$ such that $c = a^b$, we say that a, c are *conjugate* and write $a \sim c$. Prove that this is an equivalence relation, i.e.
   (1) $x \sim x$ (2) if $x \sim y$ then $y \sim x$ (3) if $x \sim y$ and $y \sim z$ then $x \sim z$