MATH 521B: Abstract Algebra
Homework 3: Due Feb. 9

1. In our definition of a group G, we assumed there is a unique group element id such that $id \circ a = a \circ id = a$ for all $a \in G$. Suppose we change just this definition, and have two (possibly different) identities, id_1 and id_2, that satisfy $id_1 \circ a = a \circ id_1 = a$ for all $a \in G$, and also $id_2 \circ a = a \circ id_2 = a$ for all $a \in G$. Prove that, in fact, $id_1 = id_2$.

2. In our definition of a group G, we assumed that there is a group element id such that $id \circ a = a \circ id = a$ for all $a \in G$. Suppose we change just this definition, and instead have two (possibly different) identities, id_L and id_R, that satisfy $id_L \circ a = a$ and $a \circ id_R = a$ for all $a \in G$. Prove that, in fact, $id_L = id_R$.

3. In our definition of a group G, we assumed that for every $a \in G$ there is a unique inverse a^{-1} such that $a \circ a^{-1} = a^{-1} \circ a = id$. Suppose we change just this definition, and instead for each $a \in G$ have two (possibly different) inverses, a_1^{-1} and a_2^{-1}, that satisfy $a_1^{-1} \circ a = a \circ a_1^{-1} = id$, and also $a_2^{-1} \circ a = a \circ a_2^{-1} = id$. Prove that for all $a \in G$, in fact, $a_1^{-1} = a_2^{-1}$.

4. In our definition of a group G, we assumed that for every $a \in G$ there is an inverse a^{-1} such that $a \circ a^{-1} = a^{-1} \circ a = id$. Suppose we change just this definition, and instead for each $a \in G$ have two (possibly different) inverses, a_L^{-1} and a_R^{-1}, that satisfy $a_L^{-1} \circ a = id = a \circ a_R^{-1}$. Prove that for all $a \in G$, in fact, $a_L^{-1} = a_R^{-1}$.

5. Let G be a group. Prove that it is *left cancellative*, i.e. for all $a, b, c \in G$, if $a \circ b = a \circ c$, then $b = c$.

6. Let G be a group. Prove that it is *right cancellative*, i.e. for all $a, b, c \in G$, if $b \circ a = c \circ a$, then $b = c$.

7. Let G be a group, and let $a, b \in G$. Prove that $(a \circ b)^{-1} = b^{-1} \circ a^{-1}$.

8. Let G be a group, and let $a \in G$. Prove that $(a^{-1})^{-1} = a$.

9. Let G be a group. Prove that $G \cong G$.

10. Let G, H be groups, and suppose that $G \cong H$. Prove that $H \cong G$.

11. Let F, G, H be groups, and suppose that $F \cong G$ and $G \cong H$. Prove that $F \cong H$.

12. Let G, H be groups, and suppose that $G \cong H$ via $f : G \to H$. Suppose that K is a subgroup of G. Prove that $f(K) = \{f(k) : k \in K\}$ is a subgroup of H.

13. Let G, H be groups, and suppose that $G \cong H$ via $f : G \to H$. For every $a \in G$, prove that $|a| = |f(a)|$, i.e. the order of a in G equals the order of $f(a)$ in H.

14. Find an example of two groups G, H such that G is a subgroup of H, $G \neq H$, and yet $G \cong H$. Hint: G, H must be infinite.