MATH 521B: Abstract Algebra
Homework 1: Due Jan. 26

An *isometry* (also called a *symmetry*) of an object A in $\mathbb{R}^n$ is a function $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfying two properties:

1. For all $v \in A$, $f(v) \in A$, and
2. For all $v, u \in \mathbb{R}^n$, $|f(v) - f(u)| = |v - u|$.

We say that f preserves A (property 1), and f preserves distances (property 2). Note: v, u above are vectors.

A key property of isometries is that they preserve structure. An isometry maps a corner of A to a corner of A, of the same degree. It maps an edge of A to an edge of A, of the same length, and the same degrees at the ends. It maps a face of A to a face of A, with the same number of sides and the same area.

1. Explicitly write down four different isometries of $A = \{(0,0)\}$ in $\mathbb{R}^2$.
   Example: $f(x, y) = (y, x)$.
2. Explicitly write down some $f : \mathbb{R}^2 \to \mathbb{R}^2$ that preserves $A = \{(0,0)\}$ but does NOT preserve distances.
3. Explicitly write down some $f : \mathbb{R}^2 \to \mathbb{R}^2$ that preserves distances but does NOT preserve $A = \{(0,0)\}$.
4. Explicitly write down all the isometries in $\mathbb{R}^2$ for A the line segment between $(-1,0)$ and $(1,0)$. Justify why your list is complete.
5. Explicitly write down all the isometries in $\mathbb{R}^2$ for A the line segment between $(-1,0)$ and $(2,0)$. Justify why your list is complete.
6. Explicitly write down all the isometries in $\mathbb{R}^2$ for A the triangle with vertices $(-1,0)$, $(1,0)$, $(0,1)$. Justify why your list is complete.
7. How many isometries are there in $\mathbb{R}^2$ for A the triangle with vertices $(-1,0)$, $(1,0)$, $(0,\sqrt{3})$. Justify why your list is complete. Why is the answer different from the previous problem?
8. Explicitly write down all the isometries in $\mathbb{R}^2$ for A the unit circle $x^2 + y^2 = 1$. Justify why your list is complete. Hint: polar coordinates.
9. Explicitly write down all the isometries in $\mathbb{R}^3$ for A the line segment between $(0,0,-1)$ and $(0,0,1)$. Justify why your list is complete. Hint: cylindrical coordinates.