Please recall the following:
\[ C^\times = \{ x \in \mathbb{C} : x \neq 0 \} \text{ and } \mathbb{R}^\times = \{ x \in \mathbb{R} : x \neq 0 \} \text{ are groups under multiplication. } \mathbb{R}, \mathbb{Q}, \mathbb{Z} \text{ are groups under addition. } S_n \text{ is the symmetric group on } [n], \text{ and } A_n \leq S_n \text{ is the alternating group on } [n] \text{ that consists of even permutations. } A \times B \text{ denotes the external direct product } \{(a, b) : a \in A, b \in B\}. \]

If \( A, B \) are groups then so is \( A \times B \) with group operation \((a, b)(a', b') = (aa', bb')\).

1. Set \( G = \{id, (1 2)(3 4), (1 3)(2 4), (1 4)(2 3)\}\). Calculate the Cayley table for \( S_4/G \).

2. Set \( G = \{id, (1 2)(3 4), (1 3)(2 4), (1 4)(2 3)\}\). Prove that \( G \leq A_4 \), and calculate the Cayley table for \( A_4/G \).

3. Fix \( H = \langle(1 2 3)\rangle \leq S_3 \). Prove that \( H \leq S_3 \), and find a familiar group isomorphic to \( S_3/H \).

For problems 4-7, let \( G = \{(a b) : a, b, d \in \mathbb{R}, ad \neq 0\} \), and \( N = \{(\frac{1}{b} 1) : b \in \mathbb{R}\} \).

4. Prove that \( N \leq G \).

5. For each \( (\frac{a}{b} \frac{c}{d}) = x \in G \), determine explicitly all \( y \in G \) with \( x \equiv y \) (mod \( N \)).

6. Prove that \( N \cong \mathbb{R} \).

7. Prove that \( G/N \cong \mathbb{R}^\times \times \mathbb{R}^\times \).

For problems 8-9, define \( U \leq C^\times \) via \( U = \{a + bi : a^2 + b^2 = 1\}\).

8. Prove that \( C^\times/U \cong \mathbb{R}^\times \).

9. Prove that \( \mathbb{R}/\mathbb{Z} \cong U \).

10. Fix a finite group \( G \), with \( N \leq G \). Set \( m = |G : N| \). Prove that \( a^m \in N \), for all \( a \in G \).

11. Fix abelian group \( G \), with \( |G| = 2k \), and \( k \) odd. Prove that \( G \) has exactly one element \( g \) with \( |g| = 2 \).

12. Let \( p \) be an odd prime, and let \( G \) be a nonabelian group with \( |G| = 2p \). Prove that \( G \) contains an element of order \( p \).

13. Fix abelian group \( G \). Set \( K = \{g \in G : |g| \leq 2\} \), \( H = \{x^2 : x \in G\} \). Prove that \( K \leq G \), \( H \leq G \), and that \( G/K \cong H \).

14. Let \( f : G \to H \) be an onto homomorphism. Suppose \( N \leq G \). Prove that \( f(N) \leq H \).

15. Fix groups \( G, H \), and suppose \( M \leq G \) and \( N \leq H \). Prove that \( (M \times N) \leq (G \times H) \).

16. Fix groups \( G, H \), and suppose \( M \leq G \) and \( N \leq H \). Prove that \( (G/M) \times (H/N) \cong (G \times H)/(M \times N) \).

17. Fix a finite group \( G \), and some \( s \in \mathbb{N} \). Set \( T = \{K : |K| = s, K \leq G\} \), the set of all subgroups of order \( s \), and assume \(|T| \geq 1\). Set \( N = \bigcap T \). Prove that \( N \leq G \).

18. Fix \( n \geq 5 \). Set \( T = \{(r s t)\} \subseteq S_n \), the set of all permutations that consist of a single cycle of length three. Prove that \( A_n = (T) \).

19. Fix \( n \geq 5 \) and \( N \leq A_n \). Set \( T = \{(r s t)\} \subseteq S_n \), the set of all permutations consisting of just a cycle of length three. If \( N \cap T \neq \emptyset \), prove that \( N = A_n \).

20. Fix a group \( G \), and suppose \( N \trianglelefteq G \). Suppose \( N \) is maximal normal, i.e. there is no \( M \) with \( N < M < G \). Prove that \( G/N \) is simple.