MATH 521A: Abstract Algebra

Homework 11 Solutions

1. Let R be a commutative ring, and let I, J be ideals of R. Prove that $I \cap J$ is an ideal of R.

First, since I, J are ideals, then $0 \in I$ and $0 \in J$. Hence $0 \in I \cap J$, so it is nonempty. Second, let $a, b \in I \cap J$. Then $a, b \in I$. Since I is an ideal, $a - b \in I$. Similarly, since $a, b \in J$, and J is an ideal, $a - b \in J$. Hence $a - b \in I \cap J$. Lastly, let $a \in I \cap J$ and $r \in R$. Since I, J are ideals, then $ra \in I$ and $ra \in J$. Combining, $ra \in I \cap J$.

2. Find I, J, ideals of $\mathbb{Z}$, such that $I \cup J$ is not be an ideal of $\mathbb{Z}$.

Many examples are possible; here is one. Set $I = (2), J = (3)$, principal ideals. We have $2 \in I \subseteq I \cup J$, and $3 \in J \subseteq I \cup J$. But their sum is $2 + 3 = 5 \notin I \cup J$, since $5 \notin I$ and $5 \notin J$. Hence $I \cup J$ is not closed under addition, and is therefore not an ideal.

3. Let R be a commutative ring, and let I, J be ideals of R. Prove that $I + J = \{a + b : a \in I, b \in J\}$ is an ideal of R.

First, since I, J are ideals, then $0 \in I$ and $0 \in J$. Hence $0 = 0 + 0 \in I + J$, so it is nonempty. Second, let $x, x' \in I + J$. Then, there are $a, a' \in I, b, b' \in J$ such that $x = a + b, x' = a' + b'$. We have $x - x' = (a + b) - (a' + b') = (a - a') + (b - b')$. Since I is an ideal, $a - a' \in I$. Since J is an ideal, $b - b' \in J$. Hence $x - x' \in I + J$. Lastly, let $x \in I + J$ and $r \in R$. There are $a \in I, b \in J$ with $x = a + b$. Since I, J are ideals, then $ra \in I$ and $ra \in J$. Hence $rx = ra + rb \in I + J$.

4. Let R be a commutative ring, and let I, J be ideals of R. Prove that $IJ = \{\sum_{i=1}^k a_i b_i : k \in \mathbb{N}, a_i \in I, b_i \in J\}$ is an ideal of R.

First, since I, J are ideals, then $0 \in I$ and $0 \in J$. Hence $0 = 0 \cdot 0 \in IJ$, so it is nonempty. Next, let $\sum_{i=1}^k a_i b_i, \sum_{i=1}^j a'_i b'_i \in IJ$. Define $a''_i = \begin{cases} a_i & i \leq k \\ -a'_{i-k} & k+1 \leq i \leq k+j \end{cases}$, and $b''_i = \begin{cases} b_i & i \leq k \\ b'_{i-k} & k+1 \leq i \leq k+j \end{cases}$ similarly. We have $a''_i \in I$ and $b''_i \in J$ by construction, and $\sum_{i=1}^k a_i b_i - \sum_{i=1}^j a'_i b'_i = \sum_{i=1}^{j+k} a''_i b''_i \in IJ$. Lastly, for any $r \in R$, we have $r \sum_{i=1}^k a_i b_i = \sum_{i=1}^k (ra_i) b_i \in IJ$, since each $ra_i \in I$.

5. Find I, J, ideals of $\mathbb{Z}[x]$, such that $K = \{ab : a \in I, b \in J\}$ is not an ideal of $\mathbb{Z}[x]$. Hint: Neither ideal can be principal.

Many examples are possible; here is one. Set $I = (2, x), J = (3, x)$. We have $3x (= x \cdot 3) \in K$, and also $2x (= 2 \cdot x) \in K$. If K were an ideal, it would also contain $3x - 2x = x$. Hence, there would be polynomials $r(x), s(x), u(x), v(x) \in \mathbb{Z}[x]$, such that $x = (2r(x) + xs(x))(3u(x) + xv(x))$. Since $\deg(x) = 1$, then either $s(x) = 0$ or $v(x) = 0$ (otherwise the degree would be at least 2). If $s(x) = 0$, then the RHS has content 2, but x has content 1, a contradiction. If $v(x) = 0$, then the RHS has content 3, but x has content 1, again a contradiction.

6. Let R be a commutative ring, and let I, J be ideals of R. Prove that $IJ \subseteq I \cap J$.
It suffices to prove that $ab \in I \cap J$, for all $a \in I, b \in J$; this is because $I \cap J$ is closed under addition. Now, $b \in J \subseteq R$; since I is an ideal, $ab \in I$. Also, $a \in I \subseteq R$; since J is an ideal, $ab \in J$. Thus $ab \in I \cap J$.

7. Let R be a commutative ring, and suppose that $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ is an infinite tower of ideals, each contained in the next. Set $I = \cup_{j=1}^{\infty} I_j$. Prove that I is an ideal.
First, since I_1 is an ideal, $0 \in I_1$, so $0 \in I$. Hence I is nonempty. Next, suppose that $a, b \in I$. There must be some $j \geq 1$ such that $a \in I_j$. There must also be some $k \geq 1$ such that $b \in I_k$. Now, choose any $m \geq \max(j, k)$. We have $a \in I_j \subseteq I_m$, and $b \in I_k \subseteq I_m$. Since I_m is an ideal, it is closed under subtraction, so $a - b \in I_m \subseteq I$. Lastly, let $a \in I$ and $r \in R$. We must have some $k \geq 1$ with $a \in I_k$. Since I_k is an ideal, $ra \in I_k \subseteq I$.

8. Find all ideals in $\mathbb{Z}_8$, and then use the first isomorphism theorem to find all homomorphic images of $\mathbb{Z}_8$.

In $\mathbb{Z}_8$, note that $3+3+3 = 1, 5+5+5+5+5 = 1, 7+7+7+7+7+7+7 = 1$; hence if an ideal contains an odd number it must contain 1, and thus all of $\mathbb{Z}_8$. If an ideal contains 2, then it contains every even number, i.e. (2). Since $6+6+6 = 2$, if an ideal contains 6, then it again is (2). The third possible ideal is $(4) = \{0, 4\}$.

Now, if all of $\mathbb{Z}_8$ is the kernel of an isomorphism, then the homomorphic image has a single element, i.e is the trivial ring $\{0\}$. If (2) is the kernel, there are two equivalence classes, so the homomorphic image has two elements, i.e. is the ring $\mathbb{Z}_2$. Lastly, if (4) is the kernel, the homomorphic image is $\mathbb{Z}_4$.

9. Prove that every ideal in $\mathbb{Z}$ is principal.
Let I be an ideal. If $I = \{0\}$, then $I = (0)$, principal. Otherwise, I contains a nonzero element x, and hence (by considering $0 - x$ if necessary) a positive element. Let y be the smallest positive element of I. Since $y \in I$, in fact $(y) \subseteq I$. Now, let $a \in I$. By the division algorithm, there are $q, r \in \mathbb{Z}$ with $a = yq + r$, and $0 \leq r < q$. Since $a, y \in I$, in fact $a - \underbrace{y + y + \cdots + y}_{q} = r \in I$. But y was the smallest positive element of I, so in fact $r = 0$. Hence $y|a$ and so $a \in (y)$. Since $a \in I$ was arbitrary, this proves $I \subseteq (y)$. Combinining, $I = (y)$.

10. Use the first isomorphism theorem to find all homomorphic images of $\mathbb{Z}$.

By the previous problem, the ideals of $\mathbb{Z}$ are just (n) for some $n \in \mathbb{Z}$. Hence these are the possible kernels of a homomorphism. For $n = 0$, we have $\mathbb{Z}/(0) \cong \mathbb{Z}$. For all other n, since $\mathbb{Z}/(n) \cong \mathbb{Z}_n$, these are exactly the homomorphic images of $\mathbb{Z}$.