MATH 521A: Abstract Algebra
Homework 1: Due Sep. 6

1. Set $S = \{-1\} \cup \mathbb{N}_0 = \{-1, 0, 1, 2, 3, \ldots\}$. Prove that S is well-ordered.
2. Suppose that $S = \{s_1, s_2, \ldots, s_k\}$ is a finite set. Prove that S is well-ordered.
3. Suppose that S and T are both well-ordered, and that $S \cap T = \emptyset$ (i.e. S, T are disjoint). Prove that $S \cup T$ is well-ordered.*
   Note: Exercises 2 and 3 together, prove (among other things) Exercise 1. However, please solve Exercise 1 directly.
4. Use the division algorithm to prove that every integer is either even or odd.
5. Use the division algorithm to prove that the square of any integer a is of the form $5k$, of the form $5k + 1$, or of the form $5k + 4$, for some integer k.
6. Prove the following variant of the division algorithm: Let a, b be integers with $b > 0$. Then there exist integers q, r such that $a = bq + r$ with $0 < r \leq b$.
7. Suppose that a, b, c are integers, with $a|b$ and $b|c$. Prove that $a|c$.
8. Determine $\gcd(n, n + 2)$ for all integers n.
   Hint: the answer depends on whether n is even or odd.

Let $a, b \in \mathbb{Z}$ with $b \neq 0$. Set $CD(a, b) = \{c \in \mathbb{Z} : c|a \text{ and } c|b\}$, the set of common divisors. Set $PLC(a, b) = \{e \in \mathbb{N} : \exists m, n \in \mathbb{Z}, e = am + bn\}$, the set of positive linear combinations.

9. Prove that $\gcd(a, b)$ is the largest element in $CD(a, b)$, and that each element of $CD(a, b)$ divides $\gcd(a, b)$.
10. Prove that $\gcd(a, b)$ is the smallest element in $PLC(a, b)$, and that $\gcd(a, b)$ divides each element of $PLC(a, b)$.*

Warning: For these problems, do not use the Fundamental Theorem of Arithmetic (unique factorization of integers into primes).

*Exercises marked with * are a bit harder.