MATH 521A: Abstract Algebra
Homework 5 Solutions

1. Let R be a ring, with $a, b \in R$. Prove that if ab is a left zero divisor, then either a or b must be a left zero divisor.

   Suppose that ab is a left zero divisor. Then $ab \neq 0$, hence $a \neq 0$ and $b \neq 0$. Also, there is some nonzero $c \in R$, with $(ab)c = 0$. By associativity, $a(bc) = 0$. If $bc \neq 0$, then a is a left zero divisor, and if $bc = 0$, then b is a left zero divisor.

2. Let R be a ring, with nonzero $a \in R$. Prove that if a is not a left zero divisor, then a may be cancelled on the left. That is, if $ab = ac$, then $b = c$.

   Suppose that $ab = ac$. We rewrite as $ab - ac = 0$, then by distributivity $a(b - c) = 0$. Since a is nonzero, and a is not a left zero divisor, then $b - c = 0$. Hence $b = c$.

3. Let R be a ring with identity, with $a \in R$. Suppose that a is a unit. Prove that multiplicative inverses are two-sided, i.e. $ab = 1$ if and only if $ba = 1$.

   Suppose $ab = 1$. Multiplying on the right by a we get $aba = 1a = a = a1$. Since a is a unit, it is not a zero divisor (by a theorem from class), hence not a left divisor. By Problem 2, we may cancel on the left, getting $ba = 1$. The other direction is similar: if $ba = 1$, we multiply by a on the left, getting $aba = a = 1a$, and then cancel a on the right to get $ab = 1$.

4. Let R be a ring with identity, with $a \in R$. Suppose that a is a unit. Prove that multiplicative inverses are unique, i.e. if $ab = 1$ and $ac = 1$, then $b = c$.

   Suppose that $ab = 1$ and $ac = 1$. By Problem 3, $ca = 1$. We multiply $ab = 1$ on the left by c, getting $c(ab) = c1 = c$. By associativity, $c(ab) = (ca)b = 1b = b$. Hence $b = c$.

5. Let R and $S \subseteq R$ both be rings with identity. Find an example where $1_S \neq 1_R$.

   Let $R = \mathbb{Z} \times \mathbb{Z}$, and $S = \{(a, 0) : a \in \mathbb{Z}\}$, a subring of R. We have $1_R = (1, 1)$ while $1_S = (1, 0)$.

6. Let R and $S \subseteq R$ both be integral domains. Prove that $1_S = 1_R$.

   Working in R, we have $1_S 1_R = 1_S$, because 1_R is neutral in R. Working in S, we have $1_S 1_S = 1_S$, because 1_S is neutral in S. But the rings have the same multiplication, so (in R), $1_S 1_R = 1_S = 1_S 1_S$. Now, 1_S is not a left zero divisor in R (since there are no zero divisors of any kind in an integral domain), so by problem 2 we may cancel on the left from $1_S 1_R = 1_S 1_S$ to conclude that $1_R = 1_S$.

7. Consider $R = \{\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) : a \in \mathbb{Z},\ b, c \in \mathbb{Q}\}$, a subring of the 2×2 matrix ring over $\mathbb{R}$. Determine the units and left zero divisors of R.

   To find units, we calculate $\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)\left(\begin{smallmatrix} a' & b' \\ 0 & c' \end{smallmatrix}\right) = \left(\begin{smallmatrix} aa' & ab'+bc' \\ 0 & cc' \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Hence a, a' are units in $\mathbb{Z}$, i.e. $a = a' = \pm 1$. Also c, c' are units in $\mathbb{Q}$, i.e. nonzero. These two necessary conditions are actually sufficient; provided that $a = \pm 1$ and $c \neq 0$, we set $a' = a, c' = \frac{1}{c}, b' = \frac{-b}{ac}$ and the result is in R.

   To find left zero divisors, we calculate $\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)\left(\begin{smallmatrix} a' & b' \\ 0 & c' \end{smallmatrix}\right) = \left(\begin{smallmatrix} aa' & ab'+bc' \\ 0 & cc' \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Since $\mathbb{Z}$ and $\mathbb{Q}$ have no (left) zero divisors, we must have $a = 0$ or $a' = 0$, and also $c = 0$ or $c' = 0$. We get four cases:

(i) $a = c = 0$. Any $b \neq 0$ leads to a zero divisor: take $a' = b' = 1$ and $c' = 0$.
(ii) $a = 0, c \neq 0$. Any b leads to a left zero divisor: take $a' = 1$ and $b' = c' = 0$.
(iii) $c = 0, a \neq 0$. Any b leads to a left zero divisor: take $a' = 0, c' = 1$, and $b' = \frac{-bc'}{a}$.
(iv) a, c both nonzero. Now $a' = c' = 0$, but also $0 = ab' + bc' = ab'$. Since $a \neq 0$, $b' = 0$, but now $a' = b' = c' = 0$, so $\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)$ is not a left zero divisor.

To sum up, the left zero divisors are just those with $ac = 0$.

8. Let R be a ring, and let $S_1, S_2, \ldots$ be infinitely many subrings of R. Prove that their mutual intersection $T = \bigcap_{i \geq 1} S_i$ is a subring of R.

First, we show $T \subseteq R$; if $t \in T$, then $t \in S_1 \subseteq R$. Hence in fact $T \subseteq S_1 \subseteq R$. Now, 0_R is in each S_i, so it is in T. Next, let $a, b \in T$. For each $i \geq 1$, we have $a, b \in S_i$; since S_i is a ring $a + b \in S_i$. Since $a + b$ is in each S_i, in fact $a + b \in T$. Similarly, for $a, b \in T$ we have $ab \in S_i$ for every i, so $ab \in T$. Lastly, if $a \in T$, then for each $i \geq 1$, $a \in S_i$. So $-a \in S_i$. Since $-a$ is in each S_i, in fact $-a \in T$.

9. Let R_1, R_2 be rings. Suppose that S_1 is a subring of R_1, and S_2 is a subring of R_2. Prove that $S_1 \times S_2$ is a subring of $R_1 \times R_2$.

First, since S_1, S_2 are subrings of R_1, R_2, in particular they are subsets. Hence if $(a, b) \in S_1 \times S_2$, then $a \in S_1 \subseteq R_1$ and $b \in S_2 \subseteq R_2$, so $(a, b) \in R_1 \times R_2$. Thus $S_1 \times S_2 \subseteq R_1 \times R_2$. Next, since S_1 is a subring of R_1, then the neutral additive element of R_1, which I call 0_1, has $0_1 \in S_1$. Similarly the neutral additive element of R_2, which I call 0_2, has $0_2 \in S_2$. Now the additive neutral element of $R_1 \times R_2$ is $(0_1, 0_2)$, and $(0_1, 0_2) \in S_1 \times S_2$. Next, let $(a, b), (a', b') \in S_1 \times S_2$. We have $(a, b) + (a', b') = (a + a', b + b')$ and $(a, b)(a', b') = (aa', bb')$. Since S_1, S_2 are each rings, they are closed, so $a + a', aa' \in S_1$ and $b + b', bb' \in S_2$. Thus $S_1 \times S_2$ is closed under addition and multiplication. Lastly, let $(a, b) \in S_1 \times S_2$. We take $-(a, b) = (-a, -b)$ (note that $(a, b) + (-a, -b) = (0_1, 0_2)$), and $-a \in S_1, -b \in S_2$. Hence $-(a, b) \in S_1 \times S_2$.

10. Let R be a ring with the property that for all $x \in R$, $x^2 = x$. Prove that each element of R is its own negative, and that R is commutative.

First, let $a \in R$ and compute $(a + a) = (a + a)^2 = a^2 + a^2 + a^2 + a^2 = a + a + a + a$. Adding $-(a + a)$ to both sides we get $0 = a + a$. This proves $-a = a$ for every $a \in R$.

Second, let $a, b \in R$ and compute $(a + b) = (a + b)^2 = a^2 + ab + ba + b^2 = a + ab + ba + b$. Adding $-(a + b)$ to both sides we get $0 = ab + ba$, which rearranges as $ab = -ba$. But since each element is its own negative, $-ba = ba$ so in fact $ab = ba$. Thus a, b commute. Since a, b were arbitrary, R is commutative.