MATH 521A: Abstract Algebra

Homework 4: Due Sep. 28

1. Let R be a ring, with additive and multiplicative neutral elements $0_R, 1_R$. Prove that $0_R, 1_R$ are unique.

2. For prime p, set $\mathbb{Z}[\sqrt{p}] = \{a + b\sqrt{p} : a, b \in \mathbb{Z}\}$. Prove that $\mathbb{Z}[\sqrt{p}]$ is a subring of $\mathbb{R}$.

3. For prime p, set $\mathbb{Q}[\sqrt{p}] = \{a + b\sqrt{p} : a, b \in \mathbb{Q}\}$. Prove that $\mathbb{Q}[\sqrt{p}]$ is a subfield of $\mathbb{R}$.

4. For $k \in \mathbb{Z}$, define object R_k, which has ground set $\mathbb{Z}$, and operations $\oplus, \odot$ defined as:

$$a \oplus b = a + b, \quad a \odot b = k$$

   Determine for which k, if any, R_k is a ring.

5. Prove or disprove: If R, S are fields, then $R \times S$ is an integral domain.

6. Define R, an object with ground set $\mathbb{Z}$, and operations $\oplus, \odot$ defined as:

$$a \oplus b = a + b - 1, \quad a \odot b = a + b - ab$$

   Prove that R is an integral domain.

7. Define R, an object with ground set $\mathbb{Z}$, and operations $\oplus, \odot$ defined as:

$$a \oplus b = a + b - 1, \quad a \odot b = ab - a - b + 2$$

   Prove that R is an integral domain.

8. Define R, an object with ground set $\mathbb{Z} \cup \{+\infty\}$, and operations $\oplus, \odot$ defined as:

$$a \oplus b = \min(a, b), \quad a \odot b = a + b$$

   Prove that R satisfies every field axiom except one, and prove that R fails to satisfy that one.